

Current Density Impedance Imaging of an Anisotropic Conductivity in a Known Conformal Class

Nicholas Hoell* Amir Moradifam[†] Adrian Nachman[‡]

February 7, 2014

Abstract

We present a procedure for recovering the conformal factor of an anisotropic conductivity matrix in a known conformal class, in a domain in \mathbb{R}^n with $n \geq 2$. The method requires one internal measurement, together with a priori knowledge of the conformal class of the conductivity matrix. This problem arises in the medical imaging modality of Current Density Impedance Imaging (CDII) and the interior data needed can be obtained using MRI-based techniques for measuring current densities (CDI) and diffusion tensors (DTI). We show that the corresponding electric potential is the unique solution of a constrained minimization problem with respect to a weighted total variation functional defined in terms of the physical measurements. Further, we show that the associated equipotential surfaces are area minimizing with respect to a Riemannian metric obtained from the data. The results are also extended to allow the presence of perfectly conducting and/or insulating inclusions.

Keywords: Anisotropic, Hybrid Problems, Interior Data, Conductivity, Diffusion Tensor Imaging, Current Density Impedance Imaging

1 Introduction

Biological tissues such as muscle or nerve fibres are known to be electrically anisotropic (see e.g. [38, 40]). In this paper, we consider the problem of recovering an anisotropic electric conductivity σ of a body Ω from measurement of one current J in the interior. Such interior data can be obtained by Current Density Imaging (CDI), a method pioneered at the

*Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 2E4. E-mail: nhoell@math.toronto.edu. The author was supported in part by a MITACS Postdoctoral Fellowship.

[†]Department of Mathematics, Michigan State University, East Lansing, MI, USA. E-mail: amir@math.msu.edu. The author was supported in part by NSERC Postdoctoral Fellowship.

[‡]Department of Mathematics and the Edward S. Rogers Sr. Department of Electrical and Computer Engineering, University of Toronto, Toronto, Ontario, Canada. E-mail: nachman@math.toronto.edu. The author is supported in part by an NSERC Discovery Grant.

University of Toronto ([18, 41]) which makes use of a Magnetic Resonance Imager (MRI) in a novel way. We also rely on the MRI-based Diffusion Tensor Imaging (DTI) method to determine the conformal class of σ , as in the new DT-CD-II method recently introduced and tested experimentally in [26, 25]. Thus, we assume that the matrix-valued conductivity function is of the form:

$$\sigma(x) = c(x)\sigma_0(x), \tag{1}$$

with $\sigma_0(x)$ known from, e.g., DTI and with the so-called ‘‘cross-property’’ factor $c(x)$ a scalar function to be determined. This assumption is motivated by a number of physical studies which have shown a linear relationship between the conductivity tensor and the diffusion tensor (see e.g. [8, 25] and further references therein).

We show that, in dimension $n \geq 2$, the cross-property factor $c(x)$ can be determined from knowledge of the current J in Ω and of the corresponding prescribed voltage f on the boundary $\partial\Omega$. In fact, the only internal data we require is the scalar function

$$a = (\sigma_0^{-1}J \cdot J)^{\frac{1}{2}} \tag{2}$$

(with σ_0^{-1} denoting the inverse of the matrix σ_0). This turns out to be the appropriate extension of the corresponding earlier result for isotropic conductivities appearing in [35], where the interior data was the magnitude $|J|$.

The method we will be presenting is based on the minimization of a weighted total variation functional defined in terms of $a(x)$ and $\sigma_0(x)$. The reader is referred to Theorem 1.3 for the precise statement.

More generally, we will show that when Ω contains perfectly conducting and/or insulating inclusions, then knowledge of a , σ_0 and f determines the location of these inclusions in all but exceptional cases, as well as the function $c(x)$, and thus also the anisotropic conductivity σ , in their complement.

1.1 Background and Motivation

Mathematical work on non-invasive determination of internal conductivity has focused largely on the classical method of Electrical Impedance Tomography (EIT). There have been major advances in the understanding of this nonlinear inverse boundary value problem (see [43] for an excellent review; in particular, see [13, 19] for recent results on recovering anisotropic conductivities in a given conformal class for the special case of admissible manifolds). It has also been shown that the EIT problem is severely ill-posed, yielding images of low resolution [16, 28].

In a new class of inverse problems, which includes the one studied here, one seeks to overcome the limitations of the reconstructions obtainable from classical boundary measurements by using data that can be measured noninvasively in the interior of the object. These are known in the literature as hybrid problems (also as coupled physics, interior data or multi-wave problems), as they couple two imaging modalities to obtain internal measurements. For overviews of such methods see [6, 21]. For imaging the electric conductivity,

there are several approaches that combine aspects of EIT with MRI: MREIT, CDII, Electric Properties Imaging (see [37, 42] for recent reviews) or with ultrasound measurements: Acousto-Electrical Tomography [44, 3, 22], Impedance-Acoustic Tomography [14].

The starting point for the method presented here is the measurement of one applied current $J(x)$ at all points x inside a bounded region Ω . We briefly recall the influential idea of [18, 41] for obtaining such interior measurements using MRI. The current J induces a magnetic field $B(x)$. The component of B parallel to the static field of the imager can be determined at any point inside Ω from the corresponding change in the phase of the measured magnetization at that location. By performing rotations of the object and repeating the experiment with the same applied current, all three components of B can be recovered, and $J(x)$ is then computed using Ampère's law:

$$J(x) = \frac{1}{\mu_0} \nabla \times B(x)$$

where μ_0 is the magnetic permeability (essentially constant in tissue). For our purposes, it is important to note that this Current Density Imaging (CDI) method works equally well in anisotropic media, as no knowledge of the conductivity is needed for the determination of the current density $J(x)$.

Inside the body being imaged the electric potential $u(x)$ corresponding to the voltage $f(x)$ on the boundary solves the following Dirichlet problem for the conductivity equation:

$$\begin{aligned} \nabla \cdot \sigma \nabla u &= 0, & x \in \Omega \subset \mathbb{R}^n \\ u|_{\partial\Omega} &= f \end{aligned} \tag{3}$$

where σ is the (generally tensorial) conductivity of the material. In the case of isotropic conductivities, (i.e. scalar σ) considered in [37, 36, 35, 34, 32] and in the absence of insulating or perfectly conducting inclusions one can replace σ in the above equation using Ohm's law $|J| = \sigma |\nabla u|$ to obtain the quasilinear, degenerate elliptic, variable coefficient 1-Laplacian equation:

$$\nabla \cdot \left(|J| \frac{\nabla u}{|\nabla u|} \right) = 0, \quad x \in \Omega. \tag{4}$$

The above equation was first introduced, with the above derivation, in the article [20], where the Neumann problem was considered and examples of non-existence and non-uniqueness were given to explain that additional data was needed for determining the conductivity. In the article [34] it was shown that equipotential surfaces, namely the level sets of $u(x)$, are minimal surfaces with respect to the conformal metric $|J|^{\frac{2}{n-1}} I_n$, with I_n the $n \times n$ identity matrix; this result was then used to treat the Cauchy problem for equation (4). It turns out that the Dirichlet problem for equation (4) can have infinitely many solutions (see [35]). This difficulty was resolved in [35], where the partial differential equation (4) was replaced by the study of the variational problem for which it is the Euler-Lagrange equation. It was shown that the solution of (3) is the unique minimizer for this problem. We recall these results in the following theorem.

Theorem 1.1. ([35]) *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with a connected $C^{1,\alpha}$ boundary, $\alpha > 0$, and let μ denote Lebesgue measure on Ω . Let $(f, |J|) \in C^{1,\alpha}(\partial\Omega) \times C^\alpha(\overline{\Omega})$ with $|J| \neq 0$ μ -a.e. be associated with an unknown conductivity $\sigma \in C^\alpha(\overline{\Omega})$. Then*

$$u_\sigma = \underset{v \in W^{1,1}(\Omega) \cap C(\overline{\Omega})}{\operatorname{argmin}} \left\{ \int_{\Omega} |J| |\nabla v| \mu(dx) : v|_{\partial\Omega} = f \text{ and } \mu(\{\nabla v = 0\}) = 0 \right\}$$

exists and is unique.

Furthermore, $\sigma = \frac{|J|}{|\nabla u_\sigma|}$ is the unique $C^\alpha(\overline{\Omega})$ scalar conductivity associated to the pair $(f, |J|)$.

A generalization of the above result was later obtained in the article [32] where the isotropic conductivity was shown to be determined from knowledge of $|J|$ on the complement of open regions on which σ may be zero (in the case of insulating inclusions) or infinite (for perfectly conducting inclusions). A further extension was recently obtained in [31] to the class of functions in $BV(\Omega)$ which is more natural for the above variational problem.

1.2 Statement of Results and Outline of the Paper

In this article we will extend the imaging method described above to the case in which the conductivity is anisotropic and known to be of the form $\sigma(x) = c(x)\sigma_0(x)$ where $c(x)$ is an unknown scalar function and σ_0 is a symmetric positive definite matrix-valued anisotropic term, assumed known.

We denote by $Mat_+(\mathbb{R}, n)$ the set of symmetric, positive-definite $n \times n$ matrices with real-valued entries. $C^\alpha(\Omega, Mat_+(\mathbb{R}, n))$ will denote the set of $Mat_+(\mathbb{R}, n)$ -valued Hölder continuous functions on Ω of order $\alpha > 0$. Similarly, $C_+^\alpha(\Omega)$ will denote the space of scalar-valued, strictly positive Hölder continuous functions of order $\alpha > 0$ on Ω . We let μ denote the Lebesgue measure on sets in Ω .

We shall first prove an anisotropic analogue to Theorem 1.1 as a prelude to the more general results accounting for inclusions. For this, we will need to precisely define the class of data that arises from physical measurements.

Definition 1.2 (First notion of admissibility). *Let Ω be a bounded domain with $C^{1,\alpha}$ boundary. A triple $(f, \sigma_0, a) \in C^{1,\alpha}(\partial\Omega) \times C^\alpha(\overline{\Omega}, Mat_+(\mathbb{R}, n)) \times C^\alpha(\overline{\Omega})$ shall be said to be admissible if there exists a $c(x) \in C_+^\alpha(\overline{\Omega})$ such that*

$$a = (\sigma_0^{-1} J \cdot J)^{\frac{1}{2}},$$

where

$$J = -c\sigma_0 \nabla u$$

is the current corresponding to the potential $u \in C^{1,\alpha}(\overline{\Omega})$ solving the following BVP

$$\begin{cases} \nabla \cdot (c\sigma_0 \nabla u) = 0, & x \in \Omega \\ u|_{\partial\Omega} = f. \end{cases} \quad (5)$$

We then have the following result.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a connected $C^{1,\alpha}$ boundary, $\alpha > 0$, and let (f, σ_0, a) be an admissible triplet as in Definition 1.2, with this same α , and with $a > 0$, $\mu - a.e.$ in Ω . Then the following convex minimization problem*

$$\operatorname{argmin}_{v|_{\partial\Omega}=f} \left\{ \int_{\Omega} a(\sigma_0 \nabla v \cdot \nabla v)^{\frac{1}{2}} d\mu : v \in BV(\Omega) \right\} \quad (6)$$

has a unique solution u_{σ} .

Furthermore, the unique $C^{\alpha}(\overline{\Omega}, \operatorname{Mat}_+(\mathbb{R}, n))$ conductivity generating the current density J while maintaining the boundary voltage f is given by $\sigma = c(x)\sigma_0(x)$ with the conformal factor c determined from the formula

$$c = \frac{a}{(\sigma_0 \nabla u_{\sigma} \cdot \nabla u_{\sigma})^{\frac{1}{2}}}.$$

In Section 2 we shall prove a weaker version of the above, Theorem 2.3, where we minimize over $W^{1,1}(\Omega) \cap C(\overline{\Omega})$ rather than $BV(\Omega)$. This is for the expository purpose of presenting the main ideas while avoiding the more technical details dealt with in the subsequent sections. The result in the form presented above will then be a special case of Theorem 1.4, in the absence of inclusions.

Following this we establish, in the remainder of section 2, the geometrical result that equipotential sets $u^{-1}(\lambda) := \{u(x) = \lambda\} \cap \overline{\Omega}$ are in fact minimal surfaces with respect to a certain Riemannian metric on Ω which is defined in terms of $\sigma_0(x)$ and $a(x)$; see Corollary 2.5.

After the above preliminary results, in Section 3 we will introduce some tools from geometric measure theory required for the proof of the main uniqueness result of the paper. We will need to work with a weighted total variation functional $\int_{\Omega} |Dv|_{\varphi}$, where the weight φ is defined in terms of a and σ_0 and where $|Dv|_{\varphi}$ is a weighted distributional gradient discussed in section 3. Most of the results presented in this section originated in the article [2]. In section 4 we formulate a more general notion of admissibility, in Definition 4.1, suitable for the presence of inclusions, which involves some technical extensions of the criteria in Definition 1.2. Our uniqueness result also requires certain natural assumptions on the regions of perfect and zero conductivity O_{∞} and O_0 , respectively, as is discussed in greater detail in that section. Further, we assume mild topological conditions on the set $S = \{a = 0\}$, and refer to equation (16) for the definition of the space $BV(\Omega, S)$. If $S = \emptyset$ the results are valid in the standard space $BV(\Omega)$ of functions with bounded variation.

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with connected $C^{1,\alpha}$ boundary, with $\alpha > 0$, and let (f, σ_0, a) be an admissible triplet generated by an unknown conductivity σ in the sense of Definition 4.1 with the same α . Then the potential u is a minimizer of the problem*

$$\min \left\{ \int_{\Omega} |Dv|_{\varphi} : v \in BV(\Omega, S) \text{ and } v|_{\partial\Omega} = f \right\}, \quad (7)$$

and if \tilde{u} is another minimizer of the above problem, then $\tilde{u} = u$ in $\Omega \setminus \{a = 0\}$. The corresponding conductivity is then

$$\sigma = \frac{a}{(\sigma_0 \nabla u \cdot \nabla u)^{\frac{1}{2}}} \sigma_0 \in C^\alpha(\overline{\Omega} \setminus \overline{Z})$$

where, in the above, Z is an open set consisting of insulators, perfect conductors, and possibly singular inclusions as discussed in Section 4.

The above summarizes the results in Theorems 4.2 and Corollary 4.3.

With this shown, we prove in Section 5 (see Theorem 5.5) that level sets of solutions of the above variational problem are not only minimal surfaces, but actually area minimizers. More precisely, they minimize the area functional

$$\mathcal{A}(\Sigma) = \int_{\Sigma} a(x) (\sigma_0 n \cdot n)^{\frac{1}{2}} dS$$

which is the area of Σ induced by the Riemannian metric determined by the data as defined in (14), (see Proposition 5.5).

Finally, technical facts on existence and uniqueness of solutions to a limiting form of the conductivity equation, as well as an equivalent variational formulation are briefly presented in Section 6. Sections 7 and 8 present conclusions and acknowledgments.

2 Anisotropic Current Density Impedance Imaging in the Absence of Inclusions.

In this section we present a simplified exposition of the main results of this paper, in order to illustrate the basic ideas used in the argument and to motivate the more general results to be presented later. We also use this section to briefly introduce some of the key geometric measure-theoretic concepts we will need and expand upon later; some excellent references thereon may be found in [11, 12, 15, 27, 33].

2.1 Existence and Uniqueness for the Variational Problem

Assume that the conductivity σ is of the form $c(x)\sigma_0(x)$ with $c(x), (\sigma_0)_{ij}(x) \in C^\alpha(\overline{\Omega})$, $\alpha > 0$, $c(x) > 0$ and σ_0 symmetric and positive-definite throughout Ω .

Throughout the paper we will be using the notation

$$(\xi, \eta)_{\sigma_0} := (\sigma_0 \xi) \cdot \eta, \quad |\xi|_{\sigma_0} := ((\sigma_0 \xi) \cdot \xi)^{\frac{1}{2}}, \quad \xi, \eta \in C^\alpha(\overline{\Omega}, \mathbb{R}^n) \quad (8)$$

to denote the inner product induced by σ_0 , and the corresponding norm, where \cdot will always be taken to denote the Euclidean dot product. In what follows ∇ denotes the usual (i.e. non-covariant) partial differentiation and we use the Einstein summation convention over

repeated upper/lower indices. We will also denote by μ the standard Lebesgue measure on Lebesgue-measurable sets.

We begin by showing that the solution u to the BVP (5) is a minimizer of a functional on Ω that is defined in terms of the internal density magnitude $|J|_{\sigma_0^{-1}}$. This generalizes the corresponding result for isotropic conductivities in [35].

Lemma 2.1. *Assume that (f, σ_0, a) is an admissible triplet in the sense of Definition 1.2 and let u be a solution to the corresponding forward problem (5). Then u is a minimizer of the functional $\mathcal{F}[\cdot]$ defined by the following*

$$\mathcal{F}[v] := \int_{\Omega} a(x) |\nabla v|_{\sigma_0} d\mu, \quad (9)$$

i.e. the relation

$$\mathcal{F}[v] \geq \mathcal{F}[u] \quad (10)$$

holds for all $v \in W^{1,1}(\Omega)$ satisfying $v|_{\partial\Omega} = f$.

Proof. Let $v \in W^{1,1}(\Omega)$. Since a comes from an admissible triple, there is exists a $c(x)$ such that $a(x)$ takes the form $a = |J|_{\sigma_0^{-1}}$ for $J = -c(x)\sigma_0\nabla u$ with u the solution of (5). Then

$$\begin{aligned} \mathcal{F}[v] &= \int_{\Omega} |J|_{\sigma_0^{-1}} |\nabla v|_{\sigma_0} d\mu \\ &= \int_{\Omega} c(x) |\nabla u|_{\sigma_0} |\nabla v|_{\sigma_0} d\mu \\ &\geq \int_{\Omega} c(x) (\nabla u, \nabla v)_{\sigma_0} d\mu \end{aligned} \quad (11)$$

$$\begin{aligned} &= \int_{\Omega} \sigma \nabla u \cdot \nabla v d\mu \\ &= \int_{\partial\Omega} f \sigma \frac{\partial u}{\partial n} dS \\ &= - \int_{\partial\Omega} f J \cdot n dS \end{aligned} \quad (12)$$

with n an outer-oriented normal to $\partial\Omega$ and where, in line (12), we have integrated by parts and applied the conductivity equation on u . We use dS for the Lebesgue surface measure on $\partial\Omega$. Equality holds in line (11) if and only if ∇u and ∇v are parallel $\mu - a.e.$ In particular, we have

$$\mathcal{F}[u] = - \int_{\partial\Omega} f J \cdot n dS$$

which, on comparing with the above, shows that u is a minimizer, as claimed. □

In order to prove the main result of this section we shall need to recall some basic notions from geometric measure theory. Firstly, by $\mathcal{H}^d(\Sigma)$ we denote the d -dimensional Hausdorff measure of a set $\Sigma \subset \Omega$ defined as

$$\mathcal{H}^d(\Sigma) := \liminf_{\delta \downarrow 0} \left\{ \sum_{j=1}^{\infty} (\text{diam} E_j)^d, \quad \bigcup_{j \in \mathbb{N}} E_j \supset \Sigma, \quad \text{diam} E_j \leq \delta \right\}$$

The super-level set of a non-negative function $u(x) \in W^{1,1}(\Omega)$, given by $E_t := \Omega \cap \{u > t\}$ has so-called *locally finite perimeter*, in the sense that the vector-valued Radon measure $\nabla \chi_{E_t}$ satisfies $\int_{\Omega} |\nabla \chi_{E_t}| < \infty$ for almost all t . For such sets we shall be concerned with the *reduced boundary*.

Definition 2.2. *The reduced boundary $\partial^* E$ of a set with locally finite perimeter is the set of points in \mathbb{R}^n for which the following hold;*

i. *For all $\epsilon > 0$ one has $\int_{B(x,\epsilon)} |\nabla \chi_E| > 0$*

ii. *The measure-theoretic outer normal $\nu(x)$ determined by*

$$\nu(x) := - \lim_{\epsilon \downarrow 0} \frac{\int_{B(x,\epsilon)} \nabla \chi_E}{\int_{B(x,\epsilon)} |\nabla \chi_E|}$$

exists, and satisfies $|\nu(x)| = 1$.

For a super-level set E_t the unit normal $\nu_t(x)$ exists \mathcal{H}^{n-1} -a.e $x \in \partial^* E_t$ (see the remarks in [35]).

We now present the main result of this section.

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with a connected $C^{1,\alpha}$ boundary, $\alpha > 0$, and let $(f, \sigma_0, a) \in C^{1,\alpha}(\partial\Omega) \times C^\alpha(\bar{\Omega}, \text{Mat}_+(\mathbb{R}, n)) \times C^\alpha(\bar{\Omega})$ be an admissible triple in the sense of Definition 1.2 with $a > 0$ μ -a.e. in Ω . Denote by $\sigma \in C^\alpha(\bar{\Omega})$ the unknown generating conductivity for this triplet and u_σ the corresponding solution to the BVP (5).*

Then u_σ is the unique solution of the following minimization problem

$$\underset{v|_{\partial\Omega}=f}{\text{argmin}} \left\{ \int_{\Omega} a(\sigma_0 \nabla v \cdot \nabla v)^{\frac{1}{2}} d\mu : v \in W^{1,1}(\Omega) \cap C(\bar{\Omega}) \right\}. \quad (13)$$

Further, the anisotropic conductivity σ is recovered from the given data by the formula $\sigma(x) = c(x)\sigma_0(x)$ with

$$c = \frac{a}{|\nabla u_\sigma|_{\sigma_0}}.$$

Proof. The proof is similar to the proof of Theorem 1.1 given in [35]. First note that since the triple (f, σ_0, a) is assumed admissible, u_σ is a solution of the minimization problem (13).

To show uniqueness, assume to the contrary that another minimizer to problem 13, say $\tilde{u} \in W^{1,1} \cap C(\bar{\Omega})$, exists. Recalling the proof of Lemma 2.1 one sees that $\nabla \tilde{u} = \lambda(x) \nabla u_\sigma$ for some non-negative λ , $\mu - a.e.$. We will show that this implies equality of the minimizers away from Lebesgue-negligible sets.

As shown in Lemma 2.2 of [35], the super-level set $E_t = \{u_\sigma > t\} \cap \Omega$ has a measure-theoretic normal $\nu_t(x) = -\frac{\nabla u_\sigma}{|\nabla u_\sigma|}$ which is continuously extendible from the reduced boundary $\partial^* E_t \cap \Omega$ to the topological boundary $\partial E_t \cap \Omega$. It then follows (using Theorem 4.11 in [15]) that, for almost all t , the region $\partial E_t \cap \Omega$ is a C^1 -hypersurface with unit normal $\nu_t(x)$. Let $\gamma(s)$ be any C^1 curve contained in a connected component of $\partial E_t \cap \Omega$. Then

$$\frac{d}{ds} \tilde{u}(\gamma(s)) = \lambda(\gamma(s)) \nabla u_\sigma(\gamma(s)) \cdot \gamma'(s) = 0.$$

Therefore \tilde{u} is constant on any connected component of $\partial E_t \cap \Omega$.

When ∂E_t is a C^1 -hypersurface, each connected component Π_t of ∂E_t intersects $\partial \Omega$. This was shown in [35] and rests on the Alexander duality theorem [30]. The fact that $\tilde{u}|_{\partial \Omega} = u_\sigma|_{\partial \Omega}$ then implies that $\tilde{u}|_{\partial E_t} = u_\sigma|_{\partial E_t}$ for almost all t . We shall now use the fact that $\nabla u_\sigma \neq 0$ μ -a.e. to show that \tilde{u} and u_σ agree on a dense subset of Ω .

Define $G := \{t \in \mathbb{R} : \tilde{u}|_{\partial E_t} = u_\sigma|_{\partial E_t}\} \subset \mathbb{R}$. As established above, the complement of G , G^c , has measure 0. Suppose, towards a contradiction, that there exists a ball $B \subset \Omega$ whose closure is contained in Ω and such that $\bar{B} \cap \{x : u_\sigma(x) \in G\} = \emptyset$. Since u_σ is continuous it must map \bar{B} to an interval $[\alpha, \beta]$ and since $|\nabla u_\sigma||_B \neq 0$ $\mu - a.e.$ we have $\alpha \neq \beta$. By construction, $[\alpha, \beta] \subset \text{Range}(u_\sigma) \setminus G$, contradicting the fact that G^c has measure zero. Thus u_σ and \tilde{u} agree on a dense subset of Ω , and since both functions are continuous, they agree on all of $\bar{\Omega}$, establishing the desired uniqueness.

Finally, with $J = -c\sigma_0 \nabla u_\sigma$ we have $a = (\sigma_0^{-1} J \cdot J)^{\frac{1}{2}} = (c^2 \sigma_0 \nabla u_\sigma \cdot \nabla u_\sigma)^{\frac{1}{2}}$. This gives the desired formula for $c(x)$. □

2.2 Equipotential Sets are Minimal Surfaces in a Riemannian metric Determined from the Data

We close this section with some interesting geometrical results about the level sets of solutions to (5). Given σ_0 and the magnitude $|J|_{\sigma_0^{-1}}$ of the current, we define a Riemannian metric on Ω and show that the level sets of the corresponding potential function have zero mean curvature in this metric. In section 5 we will prove the stronger statement that these equipotential sets are in fact area minimizing. These are generalizations to anisotropic conductivities of results proved in [36, 34] for the isotropic case.

As is customary, we denote $|A| := \det A$ for $A \in \text{Mat}(\mathbb{R}, n)$ (which should not be mistaken for the norm $|V|_{\sigma_0}$ of a vector field V , as we hope shall be clear from the context).

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with $C^{1,\alpha}$ boundary and $u \in C^{1,\alpha}(\bar{\Omega})$, $\alpha > 0$. Assume the conductivity σ is of the form $c(x)\sigma_0(x)$ for $c, \sigma_0 \in C^\alpha(\bar{\Omega})$ with*

σ_0 a known positive-definite matrix-valued function and that $|\nabla u|, c(x) > 0$ μ -a.e. where u is the potential corresponding to the conductivity σ and current density J via $J = -\sigma \nabla u$.

Define the following Riemannian metric g_{ij} on Ω :

$$g_{ij} := (|\sigma_0| |J|_{\sigma_0^{-1}}^2)^{\frac{1}{n-1}} (\sigma_0^{-1})_{ij}. \quad (14)$$

Then inside Ω one has that

$$\nabla \cdot \left(\sqrt{|g|} \frac{g^{-1} \nabla u}{|g^{-1} \nabla u|_g} \right) = 0.$$

Proof. We begin by noticing that $|\sigma_0|^{\frac{1}{n-1}} |J|_{\sigma_0^{-1}}^{\frac{2}{n-1}} \sigma_0^{-1} = c^{1+\frac{1}{n-1}-\frac{n}{n-1}} \{|\sigma|(\sigma \nabla u \cdot \nabla u)\}^{\frac{1}{n-1}} \sigma^{-1}$ whereby, with the above choice of g_{ij} one has that

$$g^{-1} = \{|\sigma|(\sigma \nabla u \cdot \nabla u)\}^{\frac{1}{1-n}} \sigma$$

Defining $m(x) := |\sigma|(\sigma \nabla u \cdot \nabla u)$ gives $|g| = \frac{m^{\frac{n}{n-1}}}{|\sigma|}$. Since $|g^{-1} \nabla u|_g^2 = \{(g^{-1} \nabla u) \cdot g(g^{-1} \nabla u)\}^2$ we have $|g^{-1} \nabla u|_g = \sqrt{(g^{-1} \nabla u) \cdot \nabla u}$. Then

$$\begin{aligned} \nabla_j \left(\sqrt{|g|} \frac{g^{ij} \nabla_i u}{|g^{-1} \nabla u|_g} \right) &= \nabla \cdot \left(\frac{m^{\frac{n+1}{2(n-1)} - \frac{1}{n-1}} \sigma \nabla u}{\sqrt{|\sigma| \sigma \nabla u \cdot \nabla u}} \right) \\ &= \nabla \cdot \left(\frac{\sqrt{m(x)} \sigma \nabla u}{\sqrt{m(x)}} \right) \end{aligned}$$

It follows from the fact that u solves the conductivity equation that

$$\nabla \cdot \left(\sqrt{|g|} \frac{g^{-1} \nabla u}{|g^{-1} \nabla u|_g} \right) = 0.$$

□

The above result immediately implies the following.

Corollary 2.5. *Suppose that u, c, σ_0 are as is in proposition (2.4). Then the level sets of u , $u^{-1}(\lambda) := \{u = \lambda\} \cap \bar{\Omega}$ are surfaces of zero mean curvature in the metric*

$$g_{ij} = (|\sigma_0| |J|_{\sigma_0^{-1}}^2)^{\frac{1}{n-1}} (\sigma_0^{-1})_{ij}.$$

Proof. As in the preceding proof of Theorem 2.3 the level sets $u^{-1}(\lambda)$ are C^1 -hypersurfaces for μ -a.e. λ . The vector $n := \frac{g^{-1} \nabla u}{|g^{-1} \nabla u|_g}$ is a unit normal in the metric g_{ij} to such a level set $u^{-1}(\lambda)$. The mean curvature of a hypersurface with unit normal n is given by $H = \text{div}_g(n)$ with div_g the metric divergence. Hence $H = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} (\sqrt{|g|} n^i)$, with n^i the components of n . We conclude from Proposition (2.4) that when u satisfies the conductivity equation, we have $H = 0$.

□

3 Preliminaries for the General Case

In this section we prepare to expand upon the results in the preceding section by considering the conductivity equation over domains which may contain insulating or perfectly conducting inclusions, i.e. regions of zero or infinite conductivity, respectively. We shall give the appropriate reformulation of the forward problem (5) in this setting. We also discuss integration by parts and coarea formulae for spaces of bounded weighted variation which will play a key role in our main general uniqueness result.

3.1 Weighted Total Variation

We start by presenting some needed preliminary results about functions of bounded weighted total variation. We will always use the notation $\chi_A(x)$ to denote the characteristic function of a set A . Often, we will abbreviate vectors and matrices in component form. In addition, as earlier, we will employ the Einstein summation convention of implied summation over repeated upper and lower indices wherever appropriate.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with connected $C^{1,\alpha}$ boundary, with $\alpha > 0$, and let a be a non-negative piecewise continuous function on Ω . While the function a is now allowed to vanish, we require that its zero set $S := \{x \in \overline{\Omega} : a(x) = 0\}$ always satisfy the following structural hypothesis

$$S := O \cup \Gamma, \tag{15}$$

where Γ is a set of measure zero with at most countably many connected components, $\mathcal{H}^{n-1}(\partial\Omega \cap S) = 0$, and where O is a disjoint union of finitely many C^1 -diffeomorphic images of the unit ball, possibly empty. These technical requirements will be helpful in the uniqueness argument.

Remark Notice that if u is a continuous function on an open set containing Γ and if Γ is a set of measure zero with at most countably many connected components, then $\overline{u(\Gamma)}$ has empty interior and this is all we require about the set Γ in the uniqueness proof.

In order to treat the possible presence of inclusions we introduce the following space of functions of bounded variation in the complement of \overline{S} :

$$BV(\Omega, S) := \{u \in L^1(\Omega) \mid \int_K |Du| < \infty, \quad \forall K \subset \overline{\Omega} \setminus \overline{S}, \quad K \text{ compact}\} \tag{16}$$

This generalizes the space $BV(\Omega)$, the space of all $L^1(\Omega)$ functions with bounded variation of the distributional gradient, i.e. those functions satisfying

$$\int_{\Omega} |Du| < \infty.$$

Let $\sigma_0 \in C^\alpha(\overline{\Omega}, \text{Mat}_+(\mathbb{R}, n))$ be a symmetric positive definite matrix with components $(\sigma_0)_{ij}$ satisfying

$$m|\xi|^2 \leq \sum_{i,j=1}^n (\sigma_0)_{ij}(x) \xi^i \xi^j \leq M|\xi|^2 \quad \forall x \in \Omega, \quad \forall \xi \in \mathbb{R}^n,$$

for constants $0 < m, M < \infty$. We then denote by $\varphi(x, \xi)$ the following function

$$\varphi(x, \xi) = a(x) \left(\sum_{i,j=1}^n (\sigma_0)_{ij} \xi^i \xi^j \right)^{\frac{1}{2}}. \quad (17)$$

For $u \in BV(\Omega, S)$ we define the *weighted total variation* of u , with respect to φ , in Ω as

$$\int_{\Omega} |Du|_{\varphi} = \sup_{B \in \mathfrak{B}_{a, \sigma_0}} \int_{\Omega} u \nabla \cdot B \, d\mu, \quad (18)$$

where

$$\mathfrak{B}_{a, \sigma_0} = \{B \in L_c^{\infty}(\Omega, \mathbb{R}^n) : \nabla \cdot B \in L^n(\Omega) \text{ and } |B|_{\sigma_0^{-1}} \leq a(x) \text{ a.e. in } \Omega\}.$$

and $L_c^{\infty}(\Omega, \mathbb{R}^n)$ is the space of vector fields of compact support in Ω whose components are in $L^{\infty}(\Omega)$. We remark that the structural hypothesis (15) ensures $\mathcal{H}^n(\partial S) = 0$ so that the integrals in (18) do not depend on the values of u inside S .

In particular, we let $P_{\varphi}(A)$ denote the φ -perimeter of the set $A \subset \Omega$ given by

$$P_{\varphi}(A) := \int_{\Omega} |D\chi_A|_{\varphi}. \quad (19)$$

We remark that if A has sufficiently smooth boundary Σ then

$$P_{\varphi}(A) = \int_{\Sigma} \varphi(x, n) dS, \quad \Sigma = \partial A \subset \bar{\Omega} \quad (20)$$

where n is a unit normal to Σ and dS is the induced Euclidean surface measure. For simplicity, we shall be using the notation $P_{\varphi}(A)$ rather than the more explicit $P_{\varphi}(A, \Omega)$ throughout the paper.

It is a straightforward consequence of the definition (18) that $\int_{\Omega} |Du|_{\varphi}$ is $L_{loc}^{\frac{n}{n-1}}(\Omega)$ -lower semi-continuous. It was shown in [2] by Amar and Bellettini that for any $u \in BV(\Omega)$, one has the following integral representation formula for the weighted total variation appearing in equation (18),

$$\int_{\Omega} |Du|_{\varphi} = \int_{\Omega} h(x, v^u) |Du| \quad (21)$$

where, in the above,

$$h(x, v^u) := (|Du| - \operatorname{ess\,sup}_{B \in \mathfrak{B}_{a, \sigma_0}} (B \cdot v^u))(x) \quad |Du| - a.e. \ x \in \Omega, \quad (22)$$

and v^u denotes the vectorial Radon-Nikodym derivative $v^u(x) = \frac{dDu}{d|Du|}$. One can verify that (21) also holds for any $u \in BV(\Omega \setminus \bar{S})$. Note that the right-hand side of equation (21) makes

sense, as v^u is $|Du|$ -measurable, and hence $h(x, v^u(x))$ is as well. In particular, it can be shown (viz. [2] Prop. 7.1) that if a and σ_0 are continuous in Ω , then one has

$$h(x, v^u) = a(x) \left(\sum_{i,j=1}^n \sigma_0^{ij} v_i^u v_j^u \right)^{1/2}, \quad |Du| - a.e. \text{ in } \Omega \quad (23)$$

for every Borel set Ω and $u \in BV(\Omega)$.

Following [1] and [5], we let

$$X := \{B \in L^\infty(\Omega, \mathbb{R}^n) : \operatorname{div} B \in L^n(\Omega)\}.$$

As proven in [5], Theorem 1.2, if ν_Ω denotes the outer unit normal vector to $\partial\Omega$, then for every $B \in X$ there exists a unique function $[B \cdot \nu_\Omega] \in L^\infty_{\mathcal{H}^{n-1}}(\partial\Omega)$ such that

$$\int_{\partial\Omega} [B \cdot \nu_\Omega] u d\mathcal{H}^{n-1} = \int_\Omega u \nabla \cdot B d\mu + \int_\Omega B \cdot \nabla u d\mu, \quad \forall u \in C^1(\bar{\Omega}). \quad (24)$$

Moreover, for $u \in BV(\Omega)$ each such $B \in L^\infty(\Omega, \mathbb{R}^n)$ with $\nabla \cdot B \in L^n(\Omega)$ gives rise to a Radon measure on Ω , denoted $(B \cdot Du)$, satisfying the following

$$\int_{\partial\Omega} [B \cdot \nu_\Omega] u d\mathcal{H}^{n-1} = \int_\Omega u \nabla \cdot B d\mu + \int_\Omega (B \cdot Du), \quad \forall u \in BV(\Omega), \quad (25)$$

We refer the interested reader to [1, 5] for a proof.

We shall need the following lemma, a proof of which follows from (25), and the fact that

$$BV(\Omega, S) \cap L^\infty(\Omega) \subset BV(\Omega)$$

which can be easily verified.

Lemma 3.1. *Let S be as defined in (15). Then*

$$\int_{\partial\Omega} [B \cdot \nu_\Omega] u d\mathcal{H}^{n-1} = \int_\Omega u \nabla \cdot B d\mu + \int_\Omega (B \cdot Du) \quad (26)$$

for all $u \in BV(\Omega, S) \cap L^\infty(\Omega)$ and $B \in X$.

We conclude with a useful coarea formula for functions of bounded weighted total variation. Details can be found in [2].

Theorem 3.2 (Generalized Coarea Formula). *Let $u \in BV(\Omega)$ and suppose $\mathcal{H}^{n-1}(\Omega \cap \{u = s\}) < \infty$ holds for all $s \in \mathbb{R}$. Then*

$$\int_\Omega |Du|_\varphi = \int_{\mathbb{R}} P_\varphi(\{u > s\}) ds,$$

where P_φ denotes the φ -perimeter defined in (19).

We note that this may, on using the representation formula (21), be recast as

$$\int_\Omega |Du|_\varphi = \int_{\mathbb{R}} \int_{\Omega \cap \partial^* \{u(x) > s\}} h(x, \nu_s) d\mathcal{H}^{n-1}(x) ds \quad (27)$$

where ν_s is a unit outer-oriented normal vector to $\Omega \cap \partial^* \{u(x) > s\}$.

3.2 Modeling Regions with Zero or Infinite Conductivity

Here we discuss how to formulate a suitable version of the conductivity equation (5) in the presence of inclusions of infinite and/or zero conductivity. Throughout the paper these inclusions will be assumed to satisfy the following conditions.

Assumption 3.3 (Hypotheses on Inclusions). *Let O_∞ be an open subset of Ω satisfying $\overline{O_\infty} \subset \Omega$, meant to model perfectly conducting inclusions, and O_0 be an open subset of Ω with $\overline{O_0} \subset \Omega$, meant to model insulating inclusions. We assume*

- i. $\overline{O_\infty} \cap \overline{O_0} = \emptyset$,
- ii. $\Omega \setminus \overline{O_\infty \cup O_0}$ is connected, and
- iii. the boundaries $\partial O_\infty, \partial O_0$ are piecewise $C^{1,\alpha}$ for $\alpha > 0$,
- iv. O_0 is a mutually disjoint union of finitely many C^1 -diffeomorphic images of the unit ball, possibly empty; if $n = 2$, O_0 has at most one such component.

Let σ^{jk} and $\tilde{\sigma}^{jk}$ be symmetric positive definite matrix functions in $\Omega \setminus \overline{O_0}$. For $k > 0$ consider the conductivity problem

$$\begin{cases} \partial_{x_j} ((k\tilde{\sigma}^{ij} - \sigma^{ij})\chi_{O_\infty} + \sigma^{ij}) \partial_{x_i} u_k = 0, & \text{in } \Omega \setminus \overline{O_0} \\ \frac{\partial u_k}{\partial \nu} = 0 & \text{on } \partial O_0, \\ u_k|_{\partial \Omega} = f. \end{cases} \quad (28)$$

The perfectly conducting inclusions occur in the limiting case $k \rightarrow \infty$. The limiting solution is the unique solution to the problem:

$$\begin{cases} \partial_{x_j} (\sigma^{ij} \partial_{x_i} u) = 0, & \text{in } \Omega \setminus \overline{O_0 \cup O_\infty} \\ \nabla u = 0, & \text{in } O_\infty \\ u|_+ = u|_-, & \text{on } \partial(O_0 \cup O_\infty) \\ \int_{\partial O_\infty^n} \sigma \frac{\partial u}{\partial \nu_n}|_+ dS = 0, & n = 1, 2, \dots \\ \frac{\partial u}{\partial \nu}|_+ = 0, & \text{on } \partial O_0 \\ u|_{\partial \Omega} = f, \end{cases} \quad (29)$$

(see the Appendix for more details), where $O_\infty = \cup_{n=1}^\infty O_\infty^n$ is a partition of O_∞ into connected components. Here, as in the rest of the paper, ν is the outward unit normal vector and the subscripts \pm indicate the limits taken from the outside and inside the inclusions, respectively.

Remark For Lipschitz continuous conductivities in any dimension $n \geq 2$, or for essentially bounded conductivities in two dimensions, the solutions of the conductivity equation satisfy the unique continuation property (see, [9] and references therein). Consequently the insulating, and the possibly perfectly conducting, inclusions are the only open sets on which the interior data $|J|_{\sigma_0^{-1}}$ may vanish identically. However, in three dimensions or higher it is possible to have a Hölder continuous σ and boundary data f that yield $u \equiv \text{constant}$ in a

proper open subset $O_s \subsetneq \Omega$, see [39, 29]. We call such regions O_s *singular inclusions*. On the other hand, we will not use Ohm's law in the classical sense inside perfect conductors: the current J inside perfectly conducting inclusions is not necessarily zero whereas $\nabla u \equiv 0$ within such regions (see [4, 24]).

4 Anisotropic Current Density Impedance Imaging in the Presence of Inclusions.

From now on we assume that $\sigma \in C^\alpha(\overline{\Omega} \setminus (O_\infty \cup O_0), \text{Mat}_+(\mathbb{R}, n))$ for $\alpha > 0$ and satisfies

$$\sigma(x) = c(x)\sigma_0(x), \quad (30)$$

where $c(x) \in C_+^\alpha(\overline{\Omega} \setminus (O_\infty \cup O_0))$ is a real, scalar-valued function, bounded away from zero and finite on $\overline{\Omega} \setminus (O_\infty \cup O_0)$ to be determined and where $\sigma_0 \in C^\alpha(\overline{\Omega}, \text{Mat}_+(\mathbb{R}, n))$ is a known symmetric, positive definite, matrix-valued function on $\overline{\Omega}$.

We will show how the shape and locations of the perfectly conducting and insulating inclusions and the conductivity σ outside of the inclusions can be determined from knowledge of the boundary voltage f , σ_0 and of

$$a = \sqrt{\sigma_0^{-1} J \cdot J} = |J|_{\sigma_0^{-1}},$$

in Ω , where J is the current density vector field generated by the voltage f at $\partial\Omega$. To formulate our results, we first need to extend the notion of *admissibility* given in Definition 1.2 to allow for inclusions.

Definition 4.1. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain with connected $C^{1,\alpha}$ boundary, $\alpha > 0$. A triplet of functions $(f, \sigma_0, a) \in C^{1,\alpha}(\partial\Omega) \times C^\alpha(\overline{\Omega}, \text{Mat}_+(\mathbb{R}, n)) \times L^2(\Omega)$ is called admissible if there exist inclusions O_0 and O_∞ satisfying Assumption 3.3, a function $c(x) \in C_+^\alpha(\overline{\Omega} \setminus (O_\infty \cup O_0))$ and a divergence free vector field J such that the following three statements hold.*

i. $a = |J|_{\sigma_0^{-1}}$ in Ω .

ii. The vector field J satisfies

$$J = \begin{cases} -\sigma \nabla u & \text{in } \Omega \setminus (\overline{O_\infty \cup O_0}). \\ 0 & \text{in } O_0 \end{cases} \quad (31)$$

where $\sigma = c\sigma_0$ and where u is the corresponding solution of (29).

iii. The set S of zeros of a satisfies

$$S \cap (\Omega \setminus \overline{O_\infty}) = O_0 \cup \overline{O_s} \cup \Gamma, \quad (32)$$

where O_s is an open set (possibly empty), Γ is a Lebesgue-negligible set with at most countably many connected components.

We are now ready to state our main uniqueness results.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain with connected $C^{1,\alpha}$ boundary and let $(f, \sigma_0, a) \in C^{1,\alpha}(\partial\Omega) \times C^\alpha(\bar{\Omega}, \text{Mat}_+(\mathbb{R}, n)) \times L^2(\Omega)$ be an admissible triplet in the sense of Definition 4.1. Define $\varphi(x, \xi) = a(x)|\xi|_{\sigma_0}$ on Ω and let $S = \{a = 0\}$. Then*

i. The potential u , solving (29), is a minimizer of the problem

$$u = \operatorname{argmin}\left\{\int_{\Omega} |Dw|_{\varphi} : w \in BV(\Omega, S) \text{ and } w|_{\partial\Omega} = f\right\}. \quad (33)$$

ii. If \bar{u} is another minimizer of the above problem, then $\bar{u} = u$ in $\Omega \setminus S$.

Proof. The proof of the first part is a slightly more technical argument as that given in the proof of Lemma 2.1. Suppose that $w \in BV(\Omega, S)$. First note that for every $x \in \Omega \setminus (\overline{O_{\infty}} \cup \overline{O_0})$ there exists $\epsilon > 0$ such that $B(x, 2\epsilon) \subset \Omega$ and

$$\int_{B(x, \epsilon)} h(x, v^w) |Dw| \geq - \int_{B(x, \epsilon)} J \cdot v^w |Dw|,$$

where J is the current density vector field described in definition (4.1). Therefore

$$h(x, v^w) \geq -J \cdot v^w, \quad |Dw| - a.e. \text{ in } \Omega \setminus (\overline{O_{\infty}} \cup \overline{O_0}).$$

Thus, on using Lemma 3.1 and the fact that the current density is divergence-free away from the inclusions we have

$$\begin{aligned} \int_{\Omega \setminus (\overline{O_0} \cup \overline{O_{\infty}})} |Dw|_{\varphi} &= \int_{\Omega \setminus (\overline{O_0} \cup \overline{O_{\infty}})} h(x, v^w) |Dw| \\ &\geq - \int_{\Omega \setminus (\overline{O_0} \cup \overline{O_{\infty}})} J \cdot v^w |Dw| \\ &= - \int_{\Omega \setminus (\overline{O_0} \cup \overline{O_{\infty}})} J \cdot Dw \\ &= - \int_{\partial\Omega \setminus (\overline{O_0} \cup \overline{O_{\infty}})} J \cdot \nu f d\mathcal{H}^{n-1} \\ &= \int_{\Omega \setminus (\overline{O_0} \cup \overline{O_{\infty}})} |Du|_{\varphi}. \end{aligned}$$

Thus u is a minimizer as claimed.

If \bar{u} is another minimizer, then the above yields

$$h(x, v^{\bar{u}}) = -J \cdot v^{\bar{u}}, \quad |D\bar{u}| - a.e. \text{ in } \Omega \setminus (\overline{O_0} \cup \overline{O_{\infty}}). \quad (34)$$

On the other hand, since a is continuous in $\Omega \setminus (\overline{O_0} \cup \overline{O_{\infty}})$ equation (23) gives

$$h(x, v^{\bar{u}}) = a(x) \left(\sum_{i,j=1}^n \sigma_0^{ij} v_i^{\bar{u}} v_j^{\bar{u}} \right)^{1/2} |D\bar{u}| - a.e. \text{ in } \Omega \setminus (\overline{O_0} \cup \overline{O_{\infty}}).$$

But then, on $\Omega \setminus (\overline{O_0 \cup O_\infty})$, we have

$$\begin{aligned}
h(x, v^{\bar{u}}) &= a(x) \left(\sum_{i,j=1}^n \sigma_0^{ij} v_i^{\bar{u}} v_j^{\bar{u}} \right)^{1/2} \\
&= c(x) |\nabla u|_{\sigma_0} |v^{\bar{u}}|_{\sigma_0} \\
&\geq c(x) |(\nabla u, v^{\bar{u}})_{\sigma_0}| \\
&\geq \sigma \nabla u \cdot v^{\bar{u}} \\
&= -J \cdot v^{\bar{u}}.
\end{aligned}$$

Thus it follows from the above and (34) that

$$\frac{J}{|J|} = \frac{\nabla \bar{u}}{|\nabla \bar{u}|} = v^{\bar{u}}, \quad |D\bar{u}| - a.e. \text{ in } \Omega \setminus (\overline{O_0 \cup O_\infty}).$$

An argument similar to that of Theorem 3.5 in [31] then allows us to conclude that $u = \bar{u}$ a.e. in Ω . \square

Once u is recovered by solving (33) it is straightforward to determine the inclusions and the conformal factor in their complement as indicated below.

Corollary 4.3. *Let (f, σ_0, a) be an admissible triplet and let σ be the corresponding unknown conductivity. Let u be the unique minimizer in Theorem 4.2. Denote the union of the zero-sets of a and $|\nabla u|$ as $S \cup \{\nabla u = 0\} =: Z \cup \Gamma$, where $Z = O_\infty \cup O_0 \cup O_s$ is open and Γ has measure zero. Then, outside Z ,*

$$\sigma = \frac{a}{|\nabla u|_{\sigma_0}} \sigma_0 \in C^\alpha(\bar{\Omega} \setminus \bar{Z}, \text{Mat}_+(\mathbb{R}, n)).$$

Remark Given a solution u to (33), one can determine if an open connected component O of Z is a perfectly conducting inclusion, an insulating inclusion, or a singular inclusion as follows:

- If $\nabla u \equiv 0$ in O and $a(x) \neq 0$ for some $x \in O$, then O is a perfectly conducting inclusion.
- If $a \equiv 0$ in O and $u \not\equiv \text{constant}$ on ∂O , then O is an insulating inclusion.
- If $a \equiv 0$ in O , $u = \text{constant}$ on ∂O , and $a \notin C^\alpha(\partial O)$, then O is not a singular inclusion.
- If $a \equiv 0$ in O , $u = \text{constant}$ on ∂O , and $a \in C^\alpha(\partial O)$, then the knowledge of the magnitude of (f, σ_0, a) is not enough to determine the type of the inclusion O .

5 Geometrical Properties of Equipotential Sets

In this section we prove the area-minimizing property of the equipotential sets $u^{-1}(\lambda) := \overline{\Omega} \cap \{u = \lambda\}$ for solutions $u(x)$ of the variational problem (33). This generalizes results in [36] and [32]. The main idea of the proof goes back to [10]. See also [17].

To begin, throughout this section for a real-valued function w , and for $\lambda \in \mathbb{R}$ and $\epsilon > 0$ we define

$$w_{\lambda,\epsilon} := \min\left\{1, \max\left\{\frac{w - \lambda}{\epsilon}, 0\right\}\right\}.$$

Lemma 5.1. *Let (f, σ_0, a) be an admissible triplet in the sense of Definition 4.1, and assume that u and J are the corresponding voltage potential and current density vector field. For any $g \in L^1(\partial\Omega)$ and $w \in BV(\Omega, S)$, define*

$$I_\varphi(w, g) := \int_\Omega |Dw|_\varphi + \int_{\partial\Omega} |J|_{\sigma_0^{-1}} |n|_{\sigma_0} |w^- - g| dS, \quad (35)$$

where w^- denotes the inner trace of w on $\partial\Omega$ and n is the normal vector on $\partial\Omega$. Then for every $\lambda \in \mathbb{R}$ and $\epsilon > 0$,

$$I_\varphi(u_{\lambda,\epsilon}, f_{\lambda,\epsilon}) \leq I_\varphi(w, f_{\lambda,\epsilon}), \quad \text{for all } w \in BV(\Omega, S).$$

Proof. Since (f, σ_0, a) is admissible, $a = |J|_{\sigma_0^{-1}}$ for $J \in (L^\infty(\Omega))^n$ with $\nabla \cdot J \equiv 0$. Hence for every $w \in BV(\Omega, S)$ it follows from (21) and Lemma 3.1 that

$$\begin{aligned} I_\varphi(w, f_{\lambda,\epsilon}) &= \int_\Omega |Dw|_\varphi + \int_{\partial\Omega} |J|_{\sigma_0^{-1}} |\nu|_{\sigma_0} |w^- - f_{\lambda,\epsilon}| d\mathcal{H}^{n-1} \\ &= \int_\Omega h(x, v^w) |Dw| + \int_{\partial\Omega} |J|_{\sigma_0^{-1}} |\nu|_{\sigma_0} |w^- - f_{\lambda,\epsilon}| d\mathcal{H}^{n-1} \\ &\geq \int_\Omega J \cdot v^w |Dw| + \int_{\partial\Omega} |J|_{\sigma_0^{-1}} |\nu|_{\sigma_0} |w^- - f_{\lambda,\epsilon}| d\mathcal{H}^{n-1} \\ &= \int_\Omega J \cdot Dw + \int_{\partial\Omega} |J|_{\sigma_0^{-1}} |\nu|_{\sigma_0} |w^- - f_{\lambda,\epsilon}| d\mathcal{H}^{n-1} \\ &= \int_{\partial\Omega} J \cdot \nu w^- dS + \int_{\partial\Omega} |J|_{\sigma_0^{-1}} |\nu|_{\sigma_0} |w^- - f_{\lambda,\epsilon}| d\mathcal{H}^{n-1} \\ &\geq \int_{\partial\Omega} J \cdot \nu w^- dS + \int_{\partial\Omega} J \cdot \nu (f_{\lambda,\epsilon} - w^-) d\mathcal{H}^{n-1} \\ &= \int_{\partial\Omega} f_{\lambda,\epsilon} J \cdot \nu d\mathcal{H}^{n-1}. \end{aligned}$$

On the other hand for $w = u_{\lambda,\epsilon}$, since $\nabla u_{\lambda,\epsilon} \cdot J \equiv |\nabla u_{\lambda,\epsilon}| |J|$, equality holds in all of the above and it follows that

$$I_\varphi(u_{\lambda,\epsilon}, f_{\lambda,\epsilon}) = \int_{\partial\Omega} f_{\lambda,\epsilon} J \cdot \nu d\mathcal{H}^{n-1}.$$

□

Lemma 5.2. *Assume that u_k is a minimizer of $I_\varphi(w, f_k)$ for $k \geq 1$, and*

$$u_k \rightarrow u \text{ in } L^1(\Omega \setminus \bar{S}), \quad f_k \rightarrow f \text{ and } u_k^- \rightarrow u^- \text{ in } L^1(\partial\Omega; \mathcal{H}^{n-1}).$$

Then

$$I_\varphi(u, f) \leq I_\varphi(w, f), \quad \text{for all } w \in BV(\Omega, S).$$

Proof. It follows from the definition (18) and a standard argument that

$$\int_{\Omega} |Du|_{\varphi} \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |Du_k|_{\varphi}.$$

Since $f_k \rightarrow f$ and $u_k^- \rightarrow u^-$ in $L^1(\partial\Omega; \mathcal{H}^{n-1})$,

$$I_\varphi(u, f) \leq \liminf_{k \rightarrow \infty} I_\varphi(u_k, f_k). \quad (36)$$

Now for every $w \in BV(\Omega, S)$, we have

$$\begin{aligned} I_\varphi(u_k, f_k) &\leq I_\varphi(w, f_k) \\ &\leq I_\varphi(w, f) + \int_{\partial\Omega} |J|_{\sigma_0^{-1}} |n|_{\sigma_0} |f - f_k| \, d\mathcal{H}^{n-1} \\ &\leq I_\varphi(w, f) + C \int_{\partial\Omega} |f - f_k| \, d\mathcal{H}^{n-1}, \end{aligned}$$

for some $C > 0$. Letting $k \rightarrow \infty$ and using (36) we obtain $I_\varphi(u, f) \leq I_\varphi(w, f)$. \square

Definition 5.3. (i) *We say that a function $u \in BV(\Omega, S)$ is φ -total variation minimizing in a set $\Omega \subset \mathbb{R}^n$ if*

$$\int_{\Omega} |Du|_{\varphi} \leq \int_{\Omega} |Dw|_{\varphi} \quad \text{for all } w \in BV(\Omega, S) \text{ with } u^- = w^- \text{ on } \partial\Omega.$$

(ii) *Similarly, we say that $E \subset \Omega$ of finite perimeter is φ -area minimizing in Ω if*

$$P_\varphi(E) \leq P_\varphi(F) \quad \text{for all } F \subset \Omega \text{ such that } \chi_E^- = \chi_F^- \text{ on } \partial\Omega.$$

We are ready to establish the main result of this section, which says that equipotential hypersurfaces of solutions to (29) are φ -area minimizing in Ω .

Theorem 5.4. *Let (f, σ_0, a) be an admissible triplet in the sense of Definition 4.1 and assume that u is the corresponding voltage potential. Then*

$$E_\lambda = \{x \in \Omega : u(x) \geq \lambda\}$$

is φ -area minimizing in Ω for every λ .

Proof. The proof is similar to that Theorem 1 in [10] and Theorem 2.4 in [17]. We write the details for the convenience of the reader.

For $\lambda \in \mathbb{R}$, let $u_1 = \max(u - \lambda, 0)$, $u_2 = u - u_1$. Let $v \in BV(\Omega, S)$ with $\text{supp}(v) \subset \Omega$. It follows from the coarea formula that

$$\begin{aligned} \int_{\Omega} |Du_1|_{\varphi} + \int_{\Omega} |Du_2|_{\varphi} &= \int_{\Omega} |Du|_{\varphi} \\ &\leq \int_{\Omega} \varphi(x, D(u+v)) \\ &\leq \int_{\Omega} |D(u_1+v)|_{\varphi} + \int_{\Omega} |Du_2|_{\varphi}. \end{aligned}$$

Hence u_1 is φ -total variation minimizing. By repeating the above argument one verifies that $u_{\lambda, \epsilon}$ is also φ -total variation minimizing. It is easy to see that for a.e. $\lambda \in \mathbb{R}$,

$$\mathcal{H}^n(\{x \in \Omega : u(x) = \lambda\}) = \mathcal{H}^{n-1}(\{x \in \partial\Omega : f(x) = \lambda\}) = 0, \quad (37)$$

and one can verify that if (37) holds, then

$$u_{\epsilon, \lambda} \rightarrow \chi_{E_{\lambda}} \text{ in } L^1_{loc}(\Omega \setminus \bar{S}), \quad \chi_{\epsilon, \lambda}^- \rightarrow \chi_{E_{\lambda}}^- \text{ in } L^1(\partial\Omega; \mathcal{H}^{n-1}). \quad (38)$$

Hence it follows from Lemma 5.1 and Lemma 5.2 that $\chi_{E_{\lambda}}$ is φ -total variation minimizing in Ω , i.e. E_{λ} is φ -area minimizing in Ω .

If λ does not satisfy (37), then let λ_k be an increasing sequence such that $\lambda_k \rightarrow \lambda$ and λ_k satisfies (37) for every k . Then

$$\chi_{E_{\lambda_k}} \rightarrow \chi_{E_{\lambda}} \text{ in } L^1_{loc}(\Omega \setminus \bar{S}), \quad \chi_{E_{\lambda_k}}^- \rightarrow \chi_{E_{\lambda}}^- \text{ in } L^1(\partial\Omega; \mathcal{H}^{n-1}),$$

as $k \rightarrow \infty$, again it follows from Lemma 5.2 that E_{λ} is φ -area minimizing in Ω . \square

We now consider the data-dependent functional

$$\mathcal{A}(\Sigma) = \int_{\Sigma} |J|_{\sigma_0^{-1}}(\sigma_0 n \cdot n)^{\frac{1}{2}} dS \quad (39)$$

for codimension one smooth hypersurfaces $\Sigma \subset \Omega \setminus \bar{S}$ having unit normal n and with dS the induced Euclidean surface measure on Σ . When Σ is a smooth boundary of a subset $A \subset \Omega$, then

$$\mathcal{A}(\Sigma) = P_{\varphi}(A),$$

with P_{φ} defined in (20).

In the next proposition we show that on hypersurfaces $\Sigma \subset \Omega \setminus \bar{S}$, this measure-theoretic perimeter agrees with the area induced by the (data-dependent) Riemannian metric introduced in formula (14).

Proposition 5.5. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. Let $\sigma_0 \in C^{1,\alpha}(\overline{\Omega}, \text{Mat}_+(\mathbb{R}, n))$ and $|J|_{\sigma_0^{-1}} \in C^\alpha(\overline{\Omega})$ with $S = \{|J|_{\sigma_0^{-1}} = 0\}$. The Riemannian metric on $\Omega \setminus \overline{S}$ given by $g_{ij} = (|\sigma_0||J|_{\sigma_0^{-1}}^2)^{\frac{1}{n-1}}(\sigma_0^{-1})_{ij}$ when restricted to an oriented codimension 1 hypersurface $\Sigma \subset \Omega \setminus \overline{S}$ induces the invariant Riemannian surface measure*

$$dS_g = |J|_{\sigma_0^{-1}}(\sigma_0 n \cdot n)^{\frac{1}{2}} dS$$

with n the outer unit normal to Σ and dS the induced Euclidean surface measure on Σ .

Proof. Denoting by dV the usual, Euclidean volume element, which in local coordinates (x^1, \dots, x^n) on Ω takes the form $dV = dx^1 \wedge \dots \wedge dx^n$. We recall that the invariant Riemannian volume on $(\Omega \setminus \overline{S}, g)$ is written locally as $dV_g = \sqrt{|g|} dV$. As before we write $g = m\sigma_0^{-1}$ for $m = (|\sigma_0||J|_{\sigma_0^{-1}}^2)^{\frac{1}{n-1}}$. Then $|g| = |\sigma_0|^{\frac{n}{n-1}-1} |J|_{\sigma_0^{-1}}^{\frac{2n}{n-1}} = m |J|_{\sigma_0^{-1}}^{\frac{2n}{n-1} - \frac{2}{n-1}}$. Thus

$$\sqrt{|g|} = |J|_{\sigma_0^{-1}} \sqrt{m}$$

We have next that $|n|_{\sigma_0} = (\sigma_0 n \cdot n)^{\frac{1}{2}}$ can be written as

$$\begin{aligned} |n|_{\sigma_0} &= (\sigma_0 n \cdot n)^{\frac{1}{2}} \\ &= \sqrt{g(mg^{-1}n, g^{-1}n)} \\ &= \sqrt{m} |g^{-1}n|_g \end{aligned}$$

The surface measure dS can be written (see e.g. [23]) as $(n \lrcorner dV) |_{\Sigma}$. Therefore

$$\begin{aligned} |J|_{\sigma_0^{-1}} |n|_{\sigma_0} dS &= \frac{\sqrt{|g|}}{\sqrt{m}} \sqrt{m} |g^{-1}n|_g (n \lrcorner dV) \\ &= (|g^{-1}n|_g n) \lrcorner dV_g \\ &= g(|g^{-1}n|_g n, N) dS_g \end{aligned} \tag{40}$$

where N is the outer unit normal to Σ in the g metric and the final equality (40) follows from Lemma 13.25 in [23]. But, when n is the unit normal to Σ , $N = \frac{g^{-1}n}{|g^{-1}n|_g}$ is the unit normal to Σ in the g metric. Then

$$\begin{aligned} |J|_{\sigma_0^{-1}} |n|_{\sigma_0} dS &= g(|g^{-1}n|_g n, N) dS_g \\ &= g(|g^{-1}n|_g n, \frac{g^{-1}n}{|g^{-1}n|_g}) dS_g \\ &= g(n, g^{-1}n) dS_g \\ &= dS_g \end{aligned} \tag{41}$$

since $g(n, g^{-1}n) = n \cdot n = 1$. This is what was to be shown. \square

In view of Theorem 5.4 and the preceding Proposition 5.5 we now have strengthened Corollary 2.5 by showing that equipotential hypersurfaces minimize the Riemannian area induced by the data-dependent metric (14).

6 Appendix: Perfectly conducting and insulating inclusions

In this appendix we derive, by a limiting procedure, the boundary value problem satisfied by potentials corresponding to conductivities which can vanish or be infinite in certain regions. These derivations slightly generalize the arguments appearing in [7].

Let $O_\infty = \cup_{j=1}^\infty O_\infty^j$ be an open subset of Ω with $\overline{O_\infty} \subset \Omega$ model the union of the connected components O_∞^j ($j = 1, 2, \dots$) of *perfectly conducting inclusions*, and let O_0 be an open subset of Ω with $\overline{O_0} \subset \Omega$ model the union of all connected *insulating inclusions*. Let χ_{O_∞} and χ_{O_0} be their corresponding characteristic functions. We assume that $\overline{O_\infty} \cap \overline{O_0} = \emptyset$, $\Omega \setminus \overline{O_\infty} \cup \overline{O_0}$ is connected, and that the boundaries $\partial O_\infty, \partial O_0$ are piecewise $C^{1,\alpha}$ for $\alpha > 0$. Let $\sigma_1 \in C^\alpha(O_\infty, \text{Mat}_+(\mathbb{R}, n))$, and $\sigma \in C^\alpha(\Omega \setminus \overline{O_0} \cup \overline{O_\infty}, \text{Mat}_+(\mathbb{R}, n))$ be matrix-valued functions such that on the respective domains of σ and σ_1

$$m|\xi|^2 \leq \sigma^{ij}\xi_i\xi_j \leq M|\xi|^2, \quad m|\xi|^2 \leq \sigma_1^{ij}\xi_i\xi_j \leq M|\xi|^2, \quad (42)$$

for constants $0 < m, M < \infty$.

Extend σ to a function on $\Omega \setminus \overline{O_0}$ and, for each $0 < k < 1$, consider the conductivity problem

$$\nabla \cdot (\chi_{O_\infty}(\frac{1}{k}\sigma_1 - \sigma) + \sigma)\nabla u = 0 \text{ on } \Omega \setminus \overline{O_0}, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial O_0, \quad \text{and} \quad u|_{\partial\Omega} = f. \quad (43)$$

The condition on ∂O_0 ensures that O_0 is insulating. The problem (43) has a unique solution $u_k \in H^1(\Omega \setminus \overline{O_0})$, which also solves

$$\begin{cases} \nabla \cdot \sigma \nabla u_k = 0, & \text{in } \Omega \setminus \overline{O_\infty \cup O_0}, \\ \nabla \cdot \sigma_1 \nabla u_k = 0, & \text{in } O_\infty, \\ u_k|_+ = u_k|_-, & \text{on } \partial O_\infty, \\ (\frac{1}{k}\sigma_1 \nabla u_k) \cdot \nu|_- = (\sigma \nabla u_k) \cdot \nu|_+, & \text{on } \partial O_\infty, \\ \frac{\partial u_k}{\partial \nu}|_+ = 0, & \text{on } \partial O_0, \\ u_k|_{\partial\Omega} = f. \end{cases} \quad (44)$$

Moreover, the solution u_k of (44) is the unique minimizer of the energy functional

$$I_k[v] = \frac{1}{2k} \int_{O_\infty} |\nabla v|_{\sigma_1}^2 dx + \frac{1}{2} \int_{\Omega \setminus \overline{O_\infty \cup O_0}} |\nabla v|_\sigma^2 dx \quad (45)$$

over maps in $H^1(\Omega \setminus \overline{O_0})$ with trace f at $\partial\Omega$. We shall show below that the limiting solution (with $k \rightarrow 0$) solves

$$\begin{cases} \nabla \cdot \sigma \nabla u_0 = 0, & \text{in } \Omega \setminus \overline{O_\infty \cup O_0}, \\ \nabla u_0 = 0, & \text{in } O_\infty, \\ u_0|_+ = u_0|_-, & \text{on } \partial O_\infty, \\ \int_{\partial O_\infty^j} (\sigma \nabla u_0) \cdot \nu|_+ ds = 0, & \text{for } j = 1, 2, \dots, \\ \frac{\partial u_0}{\partial \nu}|_+ = 0, & \text{on } \partial O_0, \\ u_0|_{\partial\Omega} = f, \end{cases} \quad (46)$$

By elliptic regularity $u_0 \in C^{1,\alpha}(\Omega \setminus \overline{O_\infty \cup O_0})$ and for any $C^{1,\alpha}$ boundary portion T of $\partial(O_\infty \cup O_0)$, $u_0 \in C^{1,\alpha}((\Omega \setminus (O_\infty \cup O_0)) \cup T)$.

Proposition 6.1. *The problem (46) has a unique solution in $H^1(\Omega \setminus \overline{O_0})$. This solution is the unique minimizer of the functional*

$$I_0[u] = \frac{1}{2} \int_{\Omega \setminus \overline{O_\infty \cup O_0}} |\nabla u|_\sigma^2 dx, \quad (47)$$

over the set $A_0 := \{u \in H^1(\Omega \setminus \overline{O_0}) : u|_{\partial\Omega} = f, \nabla u = 0 \text{ in } O_\infty\}$.

Proof: Note that A_0 is weakly closed in $H^1(\Omega \setminus \overline{O_0})$. The functional I_0 is lower semicontinuous and strictly convex and, as a consequence, has a unique minimizer u_0^* in A_0 .

First we show that u_0^* is a solution of the BVP (46). Since u_0^* minimizes (47), we have

$$0 = \int_{\Omega \setminus \overline{O_0 \cup O_\infty}} \sigma \nabla u_0^* \cdot \nabla \phi dx, \quad (48)$$

for all $\phi \in H^1(\Omega \setminus \overline{O_0})$, with $\phi|_{\partial\Omega} = 0$, and $\nabla \phi = 0$ in O_∞ . In particular, if $\phi \in H_0^1(\Omega \setminus \overline{O_0})$, we get $\int_{\Omega \setminus \overline{O_\infty \cup O_0}} (\nabla \cdot \sigma \nabla u_0^*) \phi dx = 0$ and thus u_0^* solves the conductivity equation in (46). If we choose $\phi \in H^1(\Omega \setminus \overline{O_0})$, with $\phi|_{\partial\Omega} = 0$, and $\phi \equiv 0$ in O_∞ , from Green's formula applied to (48), we get $\int_{\partial O_0} (\sigma \nabla u_0^*) \cdot \nu|_+ \phi = 0$, $\forall \phi|_{\partial O_0} \in H^{1/2}(\partial O_0)$, or equivalently, $\frac{\partial u_0^*}{\partial \nu}|_{\partial O_0} = 0$. If we choose $\phi_j \in H_0^1(\Omega \setminus \overline{O_0})$ with $\phi_j \equiv 1$ in the connected component O_∞^j of O_∞ and $\phi_j \equiv 0$ in $O_\infty \setminus O_\infty^j$, from Green's formula applied to (48) we obtain $\int_{\partial O_\infty^j} (\sigma \nabla u_0^*) \cdot \nu|_+ = 0$. \square

Next we show that the equation (46) has a unique solution u_0 and, consequently, $u_0^* = u_0|_{\Omega \setminus \overline{O_0}}$. Assume that u^1 and u^2 are two solutions and let $u = u^2 - u^1$, then $u|_{\partial\Omega} = 0$ and

$$\begin{aligned} 0 &= - \int_{\Omega \setminus \overline{O_\infty \cup O_0}} (\nabla \cdot \sigma \nabla u) u dx \\ &= - \int_{\partial\Omega} (\sigma \nabla u) \cdot \nu u ds + \int_{\partial O_0} (\sigma \nabla u) \cdot \nu|_+ u ds + \int_{\partial O_\infty} (\sigma \nabla u) \cdot \nu|_+ u ds + \int_{\Omega \setminus \overline{O_\infty \cup O_0}} |\nabla u|_\sigma^2 dx \\ &= \int_{\Omega \setminus \overline{O_\infty \cup O_0}} |\nabla u|_\sigma^2 dx. \end{aligned}$$

Thus $|\nabla u| \equiv 0$ in $\Omega \setminus \overline{O_0}$. Since $\Omega \setminus \overline{O_0}$ is connected and $u = 0$ at the boundary, we conclude uniqueness of the solution of the equations (46).

Theorem 6.2. *Let u_k and u_0 be the unique solutions of (44) respectively (46) in $H^1(\Omega \setminus \overline{O_0})$. Then $u_k \rightharpoonup u_0$ and, consequently, $I_k[u_k] \xrightarrow{k \downarrow 0^+} I_0[u_0]$.*

Proof: We show first that $\{u_k\}$ is bounded in $H^1(\Omega \setminus \overline{O_0})$ uniformly in $k \in (0, 1)$. Since

$1/k > 1$, in view of (42) there exists λ, Λ so that

$$\begin{aligned}
\frac{\lambda}{2} \|\nabla u_k\|_{L^2(\Omega \setminus \overline{O_0})}^2 &\leq \frac{1}{2} \int_{\Omega \setminus \overline{O_\infty \cup O_0}} |\nabla u_k|_\sigma^2 dx + \frac{1}{2k} \int_{O_\infty} |\nabla u_k|_{\sigma_1}^2 dx \\
&= I_k[u_k] \\
&\leq I_k[u_0] \\
&\leq \frac{\Lambda}{2} \|\nabla u_0\|_{L^2(\Omega \setminus \overline{O_0})}^2,
\end{aligned} \tag{49}$$

Thus

$$\|\nabla u_k\|_{L^2(\Omega \setminus \overline{O_0})}^2 \leq \frac{\Lambda}{\lambda} \|\nabla u_0\|_{L^2(\Omega \setminus \overline{O_0})}^2. \tag{50}$$

From (50) and the fact that $u_k|_{\partial\Omega} = f$, we see that $\{u_k\}$ is uniformly bounded in $H^1(\Omega \setminus \overline{O_0})$ and hence weakly compact. Therefore, there is a subsequence $u_k \rightharpoonup u^*$ in $H^1(\Omega \setminus \overline{O_0})$, for some u^* with trace f at $\partial\Omega$.

We will show next that u^* satisfies the equations (46), and therefore $u^* = u_0$ on $\Omega \setminus \overline{O_0}$. By the uniqueness of solutions of (46) we also conclude that the whole sequence converges to u_0 .

Since $u_k \rightharpoonup u^*$ we have that $0 = \int_{\Omega \setminus \overline{O_0 \cup O_\infty}} \sigma \nabla u_k \cdot \nabla \phi dx \rightarrow \int_{\Omega \setminus \overline{O_\infty \cup O_0}} \sigma \nabla u^* \cdot \nabla \phi dx$, for all $\phi \in C_0^\infty(\Omega \setminus \overline{O_\infty \cup O_0})$. Therefore $\nabla \cdot \sigma \nabla u^* = 0$ in $\Omega \setminus \overline{O_\infty \cup O_0}$. Further, since u_k minimizes $I_k[u_k]$ we must have $\nabla u^* = 0$ in O_∞ . To check the boundary conditions, note that, for all $\phi \in C_0^\infty(\Omega)$ with $\phi \equiv 0$ in O_∞ , we have $\int_{\partial O_0} (\sigma \nabla u_k) \cdot \nu|_+ \phi ds = 0$. Using the fact that ϕ were arbitrary, by taking the weak limit in $k \rightarrow 0$, we get $\frac{\partial u^*}{\partial \nu}|_+ = 0$ on ∂O_0 . A similar argument applied to $\phi \in C_0^\infty(\Omega)$ with $\phi \equiv 0$ in O_0 , $\phi \equiv 1$ in O_∞^j , and $\phi \equiv 0$ in $O_\infty \setminus O_\infty^j$, also shows that $\int_{\partial O_\infty^j} (\sigma \nabla u^*) \cdot \nu|_+ \phi ds = 0$. Hence u^* is the unique solution of the equation (46) on $\Omega \setminus \overline{O_0}$. Thus u_k converges weakly to the solution u_0 of (46) in $\Omega \setminus \overline{O_0}$.

7 Conclusions

We have considered the reconstruction of an anisotropic conductivity conformal to a known σ_0 when one has knowledge of the internal measurement $\sqrt{\sigma_0^{-1} J \cdot J}$, for a single current density J . Such data can be obtained by a novel combination of Current Density and Diffusion Tensor measurements. We have identified a variational problem defined in terms of the measured data and shown how to calculate the conformal factor from its unique solution. Further, we have presented a solution of the problem which allows for regions of infinite or zero conductivity. We also proved that the equipotential sets minimize the area functional corresponding to a Riemannian metric defined from the measured data.

8 Acknowledgements

The authors wish to kindly thank Mike Joy, Weijing Ma, and Nahla Elsaïd for insightful discussions on DTI and the relation between the conductivity and diffusion tensors. We also

wish to express our thanks to the anonymous referees whose comments greatly improved the organization and presentation of this article.

References

- [1] G. ALBERTI, *A Lusin type theorem for gradients*, J. Functional Analysis, 100 (1991), pp. 110–118.
- [2] M. AMAR AND G. BELLETTINI, *A notion of bounded variation depending on a metric with discontinuous coefficients*, Annales de l’I.H.P., 11 (1994), pp. 91–133.
- [3] H. AMMARI, E. BONNETIER, Y. CAPDEBOSCQ, M. TANTER, AND M. FINK, *Electrical impedance tomography by elastic deformation*, SIAM J. Appl. Math., 68 (2008), pp. 1557–1573.
- [4] H. AMMARI, J. GARNIER, H. KANG, W.-K. PARK, AND K. SOLNA, *Imaging schemes for perfectly conducting cracks*, SIAM J. Appl. Math., 71 (2011), pp. 68–91.
- [5] G. ANZELLOTTI, *Traces of bounded vector fields and the divergence theorem*, Preprint, University of Trento, (1983).
- [6] G. BAL, *Hybrid Inverse Problems and Internal Functionals*, in Inside Out II, G. Uhlmann, ed., no. 60 in MSRI Publications, Cambridge University Press, 2012.
- [7] E. S. BAO, Y. Y. LI, AND B. YIN, *Gradient estimates for the perfect conductivity problem*, Arch. Rational Mech. Anal., 193 (2009), pp. 195–226.
- [8] J. BELLIVEAU, A. DALE, J. GEORGE, D. TUCH, AND V. WEDEEN, *Conductivity tensor mapping of the human brain using diffusion tensor MRI*, Proc. Natl. Acad. Sci., 20 (1998), pp. 11697–11701.
- [9] L. BERS, F. JOHN, AND M. SCHECHTER, *Partial Differential Equations*, Wiley & Sons, New York, 1964.
- [10] E. BOMBIERI, E. D. GIORGI, AND E. GIUSTI, *Minimal cones and the Bernstein problem*, Invent. Math., 7 (1969), pp. 243–268.
- [11] L. C. EVANS AND R. GARIEPY, *Measure Theory and Fine Properties of Functions*, CRC Press, 1992.
- [12] H. FEDERER, *Geometric Measure Theory*, Springer, 1996.
- [13] D. D. S. FERREIRA, C. KENIG, M. SALO, AND G. UHLMANN, *Limiting Carleman weights and anisotropic inverse problems*, Invent. Math., 178 (2009), pp. 119–171.
- [14] B. GEBAUER AND O. SCHERZER, *Impedance-acoustic tomography*, SIAM J. Appl. Math., 69 (2008), pp. 565–576.

- [15] E. GIUSTI, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser, 1984.
- [16] D. ISAACSON, *Distinguishability of conductivities by electrical-current computed tomography*, IEEE Trans, Med. Imag, 5 (1986), pp. 91–95.
- [17] R.L. JERRARD, A. MORADIFAM, A. NACHMAN, *Existence and uniqueness of minimizers of general least gradient problems*, submitted (2013).
- [18] M. JOY, G. SCOTT, AND M. HENKELMAN, *In vivo detection of applied electric currents by magnetic resonance imaging by magnetic resonance imaging*, Magnetic Resonance Imaging, 7 (1989), pp. 89–94.
- [19] C. KENIG, M. SALO, AND G. UHLMANN, *Reconstructions from boundary measurements on admissible manifolds*, Inverse Problems and Imaging, 5 (2011), pp. 859–877.
- [20] S. KIM, O. KWON, J. SEO, AND J. YOON, *On a nonlinear partial differential equation arising in magnetic resonance electrical impedance tomography*, SIAM J. Math. Anal., (2002), pp. 511–526.
- [21] P. KUCHMENT, *Mathematics of Hybrid Imaging. A Brief Review*, in The Mathematical Legacy of Leon Ehrenpreis, I. Sabadini and D. Struppa, eds., Springer, 2012.
- [22] P. KUCHMENT AND L. KUNYANSKY, *Synthetic focusing in ultrasound modulated tomography*, Inverse Problems and Imaging, 4 (2010).
- [23] J. LEE, *Introduction to Smooth Manifolds*, Springer, 2006.
- [24] A. LIENO AND R. PIERRI, *Imaging perfectly conducting objects as support of induced currents: Kirchhoff approximation and frequency diversity*, Journal of the Optical Society of America A, 19 (2002), pp. 1308–1318.
- [25] W. MA, A. NACHMAN, N. ELSAID, M. JOY, AND T. DEMONTE, *Anisotropic impedance imaging using diffusion tensor and current density measurements*, Preprint, (2013).
- [26] W. MA, T. DEMONTE, A. NACHMAN, N. ELSAID, AND M. JOY, *Experimental Implementation of a New Method of Imaging Anisotropic Electric Conductivities*, Engineering in Medicine and Biology Society (EMBC), 35 Ann Conf. July, 2013. pp. 6437-6440, ISSN: 1557-170X
- [27] F. MAGGI, *Sets of Finite Perimeter and Geometric Variational Problems*, Cambridge University Press, 2012.
- [28] N. MANDACHE, *Exponential instability in an inverse problem for the Schrödinger equation*, Inverse Problems, 17 (2001), pp. 1435–1444.
- [29] O. MARTIO, *Counterexamples for unique continuation*, Manuscripta Math., 60 (1988), p. 21.

- [30] W. MASSEY, *Algebraic Topology: An Introduction*, Springer-Verlag, 1977.
- [31] A. MORADIFAM, A. NACHMAN, AND A. TAMASAN, *Uniqueness of minimizers of weighted least gradient problems arising in conductivity imaging*, Submitted
- [32] —, *Conductivity imaging from one interior measurement in the presence of perfectly conducting and insulating inclusions*, SIAM J. Math. Anal, 44, (2012), pp. 3969-3990
- [33] F. MORGAN, *Geometric Measure Theory*, Academic Press, 2000.
- [34] A. NACHMAN, A. TAMASAN, AND A. TIMONOV, *Conductivity imaging with a single measurement of boundary and interior data*, Inverse Problems, 23 (2007), pp. 2551–2563.
- [35] —, *Recovering the conductivity from a single measurement of interior data*, Inverse Problems, 23 (2009).
- [36] —, *Reconstruction of planar conductivities in subdomains from incomplete data*, SIAM J. Appl. Math., 70 (2010), pp. 3342–3362.
- [37] —, *Current density impedance imaging*, in Tomography and Inverse Transport Theory, G. Bal, D. Finch, P. Kuchment, J. Schotland, P. Stefanov, and G. Uhlmann, eds., no. 559 in Contemp. Math., American Mathematical Society, 2011.
- [38] P. NICHOLSON, *Specific impedance of cerebral white matter*, Experimental Neurology, 13 (1965), pp. 386–401.
- [39] A. PLIŚ, *On non-uniqueness in cauchy problem for an elliptic second order differential equation*, Bull. Acad. Pol. Sci., S. Mat., XI (1963), pp. 95–100.
- [40] B. ROTH, *The electrical conductivity of tissues*, The Biomedical Engineering Handbook; CRC Press, (2000).
- [41] G. SCOTT, M. JOY, R. ARMSTRONG, AND R. HENKELMAN, *Measurement of nonuniform current density by magnetic resonance*, IEEE Trans, Med. Imag, (1991), pp. 362–374.
- [42] J. K. SEO, D. H. KIM, J. LEE, O. I. KWON, S. Z. K. SAJIB, AND E. J. WOO, *Electrical tissue property imaging using MRI at dc and Larmor frequency*, Inverse Problems, 28 (2012).
- [43] G. UHLMANN, *Electrical impedance tomography and Calderón’s problem*, Inverse Problems, 25 (2009).
- [44] H. ZHANG AND L. V. WANG, *Acousto-electric tomography*, Photons Plus Ultrasound: Imaging and Sensing (A. A. Oraevsky and L. V. Wang, Eds.), 5320 (2004), pp. 145–149.