Inverse Problems and Imaging doi:10.3934/ipi.2014.8.xx
Volume 8, No. 1, 2014, X–XX

RAY TRANSFORMS ON A CONFORMAL CLASS OF CURVES

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(Communicated by the associate editor name)

Abstract. We introduce a technique for recovering a sufficiently smooth function from its ray transforms over rotationally related curves in the unit disc of 2-dimensional Euclidean space. The method is based on a complexification of the underlying vector fields defining the initial transport and inversion formulae are then given in a unified form. The method is used to analyze the attenuated ray transform in the same setting.

1. Introduction.

In several engineering applications one deals with the problem of recovering an unknown function from its integrals over a collection of lines. In medical imaging this problem arises in positron emission tomography (PET), single photon emission tomography (SPECT), and (originally) CT-scan tomography [25]. In other applications the line integral is instead taken over a class of one-dimensional curves in either Euclidean space or more generally, a smooth manifold. Examples of this more general geometry arise, for instance, in the geophysical problem of determining internal properties of the Earth from travel-time measurements made at the surface; [34]. Another example of the general setting is in non-destructive electrical imaging techniques such as electrical impedance tomography; see e.g. [11, 9]. This type of data is generally referred to as a *ray transform* or, in the case of straight lines, the *x-ray transform*. Quite often the physics will also dictate that the signal undergoes some absorption along its trajectory and is attenuated, the data then called the *attenuated ray transform*.

The mathematical applications, properties, and uses of these integral transforms and their inverses are discussed in great detail in [11, 18, 19, 34] and include harmonic analysis, algebraic curves, tensor geometry, and partial differential equations to name a few. Generally, *explicit* inversion formulae over curves other than lines (geodesics of a Riemannian manifold, say) tend to restrict focus to manifolds with a strong amount of symmetry (as in, e.g. [18, 19, 20, 9, 31, 28]) and do not include the effects of absorption encountered during propagation. For the case where attenuation is taken into account, strong local injectivity results were established

2010 *Mathematics Subject Classification.* Primary: 35F45, 35R30; Secondary: 53C65.

Key words and phrases. Transport equation, quasiconformal, harmonic, attenuated ray transform, Beltrami equation, filtered backprojection.

The first author was partly supported by NSF grant DGE-0221041 during the preparation of this article.

by Finch in 1986 [13]. Full injectivity was only established as recently as 1998, by Arbuzov, Bukhgeim, and Kazantsev in the work [4] using the theory of A-analytic functions developed by Alexander Bukhgeim. An excellent account of this and other progress made on the attenuated ray transform may be found in Finch's chapter of the book [35]. More recently Salo and Uhlmann, in the article [32], developed a reconstruction procedure for the attenuated ray transform on geodesics of compact, two-dimensional, Riemannian manifolds with boundary, although an explicit inversion formula was not obtained. We will be, in this paper, restricting our attention to particular ray transforms on curves in a 2-dimensional region of Euclidean space.

The method we present in this paper generalizes a technique that was first used by R.G. Novikov in [27] for lines in Euclidean space and later generalized in [6] for geodesic rays in hyperbolic geometry, giving an explicit filtered backprojection inversion formula for the attenuated ray transform in each case. In fact, both of these inversion formulae are special cases of the main result of this article, Theorem 5.2, as we shall show in section 6. The technique rests on a particular complexification of a class of differential operators in $\mathbb{R}^2$ which allows us to recast the problem in terms of complex analysis in the unit disc. Once the problem is cast in this light, we use the classical Poisson formula [14] relating the boundary values of analytic functions on the unit disc to their interior values to obtain a reconstruction formula. We will also see that the method illustrates a deep role played by Beltrami equations in inverting ray transforms. Excellent introductions to complex analysis and conformal mappings are [33, 16, 14] and the classic [1]. Good introductions to quasiconformal mappings, Beltrami equations and their generalizations can be found in [2, 29] and the more recent [5]. References on Blaschke products and multivalent mappings can be found in [10, 15].

An outline of the paper is as follows. In section 2, the general setup, notation, and a quick review of the essential mathematical objects used throughout the paper are presented, together with the main results. In section 3 we begin the complexification procedure by introducing a new complex parameter λ into the transport equation introduced in section 2 and then give a classification of the vector fields under consideration as those of **type H**. Much of the heavy lifting is done in the more technical section 4 where we find and analyze the Green's function of the new parameterized complex partial differential transport equation. We will establish that **condition H** is sufficient to guarantee holomorphicity of the solution of this equation in terms of the new parameter λ. We evaluate the asymptotics of the solution as our complex parameter λ tends to the unit circle from both inside and outside, i.e. as $|\lambda| \to 1^{\mp}$ and see that in fact its imaginary part depends on the data we are interested in. Once this is established, we use this fact in section 5 to give our desired reconstruction formula in the non-attenuated case. The rest of section 5 uses the non-attenuated formula to give an integrating factor solution for the attenuated case. In section 6 examples of the method are then given for the cases of Euclidean space, the Poincaré disc and, with an easy generalization of the technique, the projection of the spherical cap into the unit disc. We offer brief concluding remarks in section 7.

A Review of the Case of Straight Lines. In order to motivate some of the general ideas presented in later sections, and to help keep the discussion concrete, we start by quickly showing how to invert the attenuated ray transform on lines in Euclidean space. The method we present in this section is similar to that discussed in the paper [7] and is a variation of the method used by R.G. Novikov in the article [27].

Physical Background. The attenuated ray transform on straight lines in Euclidean space arises naturally in the context of emission tomography in medical imaging where one measures radiation being emitted by an internal source $f(\mathbf{x})$. Letting $u(\mathbf{x},\theta)$ denote the density of light at the position $\mathbf{x} \in \Omega \subset \mathbb{R}^2$ with orientation θ, we will be considering the following stationary radiative transport equation

$$\boldsymbol{\theta} \cdot \nabla_{\mathbf{x}} u(\mathbf{x},\theta) + a(\mathbf{x})u(\mathbf{x},\theta) = f(\mathbf{x}) \tag{1}$$

where $\boldsymbol{\theta} \doteq (\cos\theta, \sin\theta)$, Ω is convex and both absorption $a(\mathbf{x})$ and density $f(\mathbf{x})$ are sufficiently smooth and compactly supported. We will always assume that the transmission coefficient $a(\mathbf{x})$ is known since this can be determined by additional measurements. Although it is not necessary, for simplicity in keeping with the rest of this article we will also assume that $\Omega \subset D^+$ where D^+ is the unit disc $\{(x,y) \in \mathbb{R}^2; x^2+y^2<1\}$.

Since light is being emitted internally, the emission event $u(\mathbf{x},\theta)$ should satisfy the boundary condition that no radiation is picked up from the behind the emission site, i.e.

$$\lim_{t \searrow -\infty} u(\mathbf{x}(t),\theta) = 0 \tag{2}$$

where $\mathbf{x}(t)$ are the characteristic curves, $\mathbf{x}+t\boldsymbol{\theta}$, of $\boldsymbol{\theta}\cdot\nabla$.

Defining $X_\theta \doteq \boldsymbol{\theta}\cdot\nabla$ and using the decay of the absorption coefficient we see that $\int_0^\infty X_\theta a(\mathbf{x}+t\boldsymbol{\theta})dt = \int_0^\infty \frac{\partial}{\partial t} a(\mathbf{x}(t))dt$ and thus $\boldsymbol{\theta}\cdot\nabla(B_\theta a)(\mathbf{x},\theta) = -a(\mathbf{x})$ for

$$(B_\theta a)(\mathbf{x},\theta) \doteq \int_0^\infty a(\mathbf{x}+t\theta)dt$$

the so-called divergent beam transform [24].

We will be using here the *symmetrized beam transform* D_θ, based on the usual divergent beam transform and defined as the odd part of the integration over characteristics with respect to angle; namely, $D_\theta \doteq \frac{1}{2}(B_{-\theta} - B_\theta)$ or

$$(D_\theta a)(\mathbf{x}) = \frac{1}{2}\int_{\mathbb{R}} sgn(t) a(\mathbf{x}-t\boldsymbol{\theta})dt \tag{3}$$

It is clear then that the symmetrized beam transform inverts the vector field X_θ, in the sense that $X_\theta D_\theta a = a$ and therefore $D_\theta a$ serves as an integrating factor in equation (1) so that we may write the solution in the following form

$$u(\mathbf{x},\theta) = e^{-D_\theta a}\int_0^\infty (e^{D_\theta a} f)(\mathbf{x}-t\boldsymbol{\theta})dt$$

With $\boldsymbol{\theta}^\perp = (-\sin\theta, \cos\theta)$ one can write $\mathbf{x} = t\boldsymbol{\theta} + s\boldsymbol{\theta}^\perp$ and we see that

$$\begin{aligned} \lim_{t \nearrow \infty} e^{D_\theta a(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta})} u(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta}) &= \int_{\mathbb{R}} e^{D_\theta a(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta})} f(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta})dt \\ &= (I_{a,\theta} f)(s,\theta) \end{aligned} \tag{4}$$

where $I_{a,\theta}f$ is the *attenuated ray transform* of f. We can acquire the measurements appearing on the left-hand side of equation (4) since a was assumed to be known and $\lim_{t\nearrow\infty} u(s\boldsymbol{\theta}^\perp + t\boldsymbol{\theta})$ is the radiation measured by external detectors.

The Inversion Method. Our method of approach for finding an inversion formula for the attenuated ray transform will be based on finding an appropriate holomorphic integrating factor for the transport equation (1). To do this, we will first find an analytic solution to the non-attenuated transport equation in an appropriate extension of the angular variable.

In order to arrive at a suitable inversion formula for the non-attenuated ray transform, we first consider the parameterization of $\mathbb{R}^2 \cong \mathbb{C}$ via $z(t,s) \doteq t + is$. Then the pushforward in z of the vector field $\frac{\partial}{\partial t}$ takes the form $z_* \frac{\partial}{\partial t} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$ and one sees that $t(z) = \frac{z+\bar{z}}{2}$ and $s(z) = \frac{z-\bar{z}}{2i}$. For each $\theta \in (0, 2\pi]$ define a conformal map $e^{i\theta} : \mathbb{C}^2 \to \mathbb{C}^2$ via $e^{i\theta} : (z, \bar{z}) \mapsto (e^{i\theta} z, e^{-i\theta} \bar{z})$.

A calculation reveals that

$$X_\theta = e^{i\theta}_* z_* \frac{\partial}{\partial t}$$

which is the vector field defining the lines over which our data is integrated.

We will begin by examining a particular complexification of the non-attenuated radiative transport equation $X_\theta u(\mathbf{x}, \theta) = f(\mathbf{x})$. The main idea will be to turn the transport equation into an elliptic equation, namely the $\bar{\partial}$-equation, in particular coordinates. For this we consider the extension of the map $e^{i\theta}$ to $\theta \in \mathbb{C} \setminus \mathbb{R}$. Define, for $0 < |\lambda| < 1$, the map $\lambda : (z, \bar{z}) \to (\lambda z, \frac{\bar{z}}{\lambda})$ generating the complexified transport operator $X_\lambda = \lambda_* z_* \frac{\partial}{\partial t}$. Since $\lambda_*(z, \bar{z}) \mapsto (\frac{z}{\lambda}, \bar{z}\lambda)$ one has that

$$X_\lambda = \lambda \frac{\partial}{\partial z} + \frac{1}{\lambda} \frac{\partial}{\partial \bar{z}}, \qquad \lambda \in D^+ \setminus \{0\}$$

To solve the complexified transport equation $X_\lambda u(z, \lambda) = f(z)$ for $\lambda \in D^+ \setminus \{0\}$ one can explicitly find the Green's function associated to X_λ by a change of variables in $\lambda_* s = s(z, \lambda)$. It turns out that this will reduce the problem to a $\bar{\partial}$-problem in s. To see this, observe that since $s(z, \lambda) = \frac{1}{2i}(\frac{z}{\lambda} - \bar{z}\lambda)$, we can, after a change of variable in $\lambda_* s$, write the following

$$\begin{aligned}(\lambda_* s)_* X_\lambda &= \frac{\lambda}{2i}\left(\frac{1}{\lambda}\frac{\partial}{\partial s} + \bar{\lambda}\frac{\partial}{\partial \bar{s}}\right) + \frac{1}{2i\lambda}\left(-\lambda\frac{\partial}{\partial s} - \frac{1}{\bar{\lambda}}\frac{\partial}{\partial \bar{s}}\right) \\ &= \frac{|\lambda|^2 - \frac{1}{|\lambda|^2}}{2i}\frac{\partial}{\partial \bar{s}} \end{aligned} \tag{5}$$

Since, on that region, the Jacobian of $s(z, \lambda)$ is $-\frac{1}{4}(|\lambda|^2 - \frac{1}{|\lambda|^2})$, we see that the fundamental equation $X_\lambda G_\lambda(z) = \delta(z)$ may be written in $(s, \bar{s})$ variables as

$$\frac{1}{i}\frac{\partial}{\partial \bar{s}}(s(z,\lambda)_* G_\lambda) = -\frac{\delta(s(z,\lambda))}{2}$$

Therefore we may write

$$G_\lambda(z) = \frac{1}{2\pi i s(z, \lambda)}, \qquad \lambda \in D^+ \setminus \{0\} \tag{6}$$

since $\bar{\partial}_z \frac{1}{\pi z} = \delta(z)$.

For $\lambda \neq 0$, the solution to $X_\lambda u(z, \lambda) = f(z)$ may therefore be written as the covolution $u(z, \lambda) = \int_{D^+} G_\lambda(z - z_0) f(z_0) d\mu(z_0)$. Since $\partial_{\bar{\lambda}} G_\lambda(z) = 0$ away from zero, we see that $u(z, \lambda)$ defined this way is holomorphic away from the origin. We remark that this solution may be analytically extended to a function (which we denote by u also) asymptotically at $\lambda = 0$ since for all $z \in D^+$, $s(z, \lambda)$ is meromorphic in λ and $\lim_{\lambda \to 0} |s(z, \lambda)| = \infty$. The solution $u(z, \lambda)$ therefore continues to a holomorphic function of λ vanishing at $\lambda = 0$. That is to say, we have an expansion of the form $u(z, \lambda) = \lambda \partial_\lambda u(z, 0) + O(|\lambda|^2)$. Then from the expansion $\partial_\lambda u(z, \lambda) = -(\frac{z}{\lambda^2} +$

$\bar{z}\lambda^2)u_z(z,\lambda)$ near $|\lambda|=0$ and the fact that by inspection $u_z(z,\lambda)=O(\lambda)$ near 0, we see that $\lambda\frac{\partial u(z,\lambda)}{\partial z}$ is necessarily an analytic function vanishing at the origin as well. Since $u_{\bar{z}}(z,\lambda)$ decays at the origin $\frac{1}{\lambda}u_{\bar{z}}(z,\lambda)$ is complex-analytic on D^+ as well.

In the limit $|\lambda|\to 1^-$ we see, on using $\lambda_\varepsilon=(1-\varepsilon)e^{i\theta}$ in equation (6), that

$$\begin{aligned} G_{\lambda_\varepsilon}(z) &= \frac{1}{\pi(ze^{-i\theta}-\bar{z}e^{i\theta}+\varepsilon(ze^{-i\theta}+\bar{z}e^{i\theta})+O(\varepsilon^2))} \\ &= \frac{1}{2i\pi(\mathbf{x}\cdot\boldsymbol{\theta}^\perp - i\varepsilon\mathbf{x}\cdot\boldsymbol{\theta}+O(\varepsilon^2))} \end{aligned}$$

Therefore, since $u(z,\lambda)$ is given by the convolution $(G_\lambda * f)(z)$, in the limit, the solution $u_+(z,\theta)=\lim_{|\lambda|\nearrow 1}u(z,\lambda)$ tends towards a convolution of $f(z(t,s))$ with the distribution $\frac{1}{2\pi i(s-i0sgn(t))}$ in both t and s. We remark that the kernel $G_+(z,\theta)=\frac{1}{2\pi i(\mathbf{x}\cdot\boldsymbol{\theta}^\perp - i0sgn(\mathbf{x}\cdot\boldsymbol{\theta}))}$ is the same as that appearing in [27]. One can show, using the Plemelj formula, that this convolution with $\frac{1}{2\pi i(s-i0sgn(t))}$ then tends towards

$$u_+(z,\theta)=(D_\theta f)(\mathbf{x})+\frac{1}{2i}(HI_\theta f)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp,\theta)$$

where H is the Hilbert transform taken with respect to the s variable.

As an aside, we mention that if λ is instead taken to be $1<|\lambda|<\infty$, the Jacobian of s changes sign and the problem may be considered on both the inside and the outside of the unit disc. In that case one sees the asymptotic limits from both inside and outside the unit disc become

$$G_\pm(z,\theta)=\frac{\pm 1}{2\pi i(\mathbf{x}\cdot\boldsymbol{\theta}^\perp \mp i0sgn(\mathbf{x}\cdot\boldsymbol{\theta}))}$$

and one can analyze the problem as a Riemann-Hilbert one. This amounts to finding a sectionally-analyic function $\psi(z,\lambda)$ away from the unit circle with jump across the unit circle given by $\phi(z,\lambda)=i(HI_\theta f)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp,\theta)$ augmented by asymptotic vanishing $\lim_{|\lambda|\nearrow\infty}|\lambda|\psi(z,\lambda)=0$. Throughout this article we will occasionally adopt notation of the Riemann-Hilbert formalism, though we will only be concerned with the interior of the unit disc.

Noticing that $\lambda_*\boldsymbol{\theta}^\perp\cdot\nabla=-i(-\lambda\frac{\partial}{\partial z}+\frac{1}{\lambda}\frac{\partial}{\partial\bar{z}})$ is the complexification of the vector field generating the flow orthogonal to X_θ, we remark that since the analytic function $\lambda\frac{\partial u(z,\lambda)}{\partial z}\to 0$ as $|\lambda|\searrow 0$ we see that we may use this to equate the limiting longitudinal and transverse flow at a point, namely

$$\begin{aligned} \lim_{|\lambda|\to 0}(\lambda\frac{\partial u(z,\lambda)}{\partial z}+\frac{1}{\lambda}\frac{\partial u(z,\lambda)}{\partial\bar{z}}) &= \lim_{|\lambda|\to 0}(-\lambda\frac{\partial u(z,\lambda)}{\partial z}+\frac{1}{\lambda}\frac{\partial u(z,\lambda)}{\partial\bar{z}}) \\ &= i\lim_{|\lambda|\to 0}\{(\lambda_*\boldsymbol{\theta}^\perp\cdot\nabla)u(\mathbf{x},\lambda)\} \end{aligned}$$

Since the left and the right sides of the above represent analytic functions, their values at zero are determined by their mean values and we may write

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi}X_\theta u_+(z,\theta)d\theta &= \frac{1}{4\pi}\int_0^{2\pi}\boldsymbol{\theta}^\perp\cdot\nabla HIf(s(ze^{-i\theta}),e^{i\theta})d\theta \\ &\quad+\frac{i}{2\pi}\int_0^{2\pi}\boldsymbol{\theta}^\perp\cdot\nabla(D_\theta f)(z)d\theta \end{aligned} \tag{7}$$

Then, comparing the real and imaginary parts above, and using that $X_\theta D_\theta f = f$ one has the formula

$$f(\mathbf{x}) = \frac{1}{4\pi}\int_0^{2\pi} \boldsymbol{\theta}^\perp \cdot \nabla\{HIf(s(\mathbf{x}\cdot\boldsymbol{\theta}^\perp), e^{i\theta})\}d\theta \tag{8}$$

which determines $f \in C_c^\infty(D^+)$ from measurements of $(If)(\mathbf{x}\cdot\boldsymbol{\theta}^\perp, \theta)$. The operator $\lambda_*\boldsymbol{\theta}^\perp \cdot \nabla$ plays the role of $-X_\lambda$ which will make appearances throughout the remainder of the article.

Next, we will consider the complexified transport equation with attenuation $a(\mathbf{x}) \in C_c^\infty(D^+)$ given by

$$X_\lambda w(z,\lambda) + a(z)w(z,\lambda) = f(z), \qquad \lambda \in D^+ \setminus \{0\} \tag{9}$$

We use the integrating factor $e^{h(z,\lambda)}$ where $X_\lambda h(z,\lambda) = a(z)$ to reduce equation (9) to

$$X_\lambda\{e^{h(z,\lambda)}w(z,\lambda)\} = e^h(z,\lambda)f(z) \tag{10}$$

and by the results previously mentioned the integrating factor will be, for all $z \in D^+$, holomorphic in $\lambda \in D^+$. Therefore, the right hand side of the above is analytic in λ and since the Green's function associated to X_λ preserves holomorphicity, one has holomorphic solutions of (9) vanishing at the origin. Since $w(z,\lambda) \to 0$ as $|\lambda| \to 0$ we have that solutions to (10) satisfy

$$\begin{aligned}\lim_{|\lambda|\searrow 0}\{X_\lambda w(z,\lambda) + a(z)w(z,\lambda)\} &= i\lim_{|\lambda|\searrow 0}(\{\lambda_*\boldsymbol{\theta}^\perp\cdot\nabla\}w(z,\lambda))\\ &= \frac{i}{2\pi}\int_0^{2\pi}\boldsymbol{\theta}^\perp\cdot\nabla w_+(z,\theta)d\theta\end{aligned} \tag{11}$$

Since the dependence involves some more complicated operators arising from the limiting values of

$$w(z,\lambda) = e^{-h(z,\lambda)}\int_{\mathbb{C}} G_\lambda(z-z_0)e^{h(z_0,\lambda)}f(z_0)d\mu(z_0)$$

as $\lambda \to T$ non-tangentially, we merely state that a careful study of $w_+(z,\theta)$ shows that it in fact depends on the data $I_{a,\theta}f$ and we refer the interested reader to equation (32) for the details.

With the above considerations in mind, the relation

$$f(\mathbf{x}) = \frac{i}{2\pi}\int_0^{2\pi}\boldsymbol{\theta}^\perp\cdot\nabla w_+(\mathbf{x},\theta)d\theta \tag{12}$$

can be shown to give our desired filtered backprojection inversion formula for the attenuated ray transform on straight lines in Euclidean space.

2. **Preliminaries.** We now proceed, keeping the preceding Euclidean example as intuition, to the general focus of this article. We let $\boldsymbol{\gamma} : \mathbb{R}^2 \ni (t,s) \mapsto \boldsymbol{\gamma}(t,s) \in \Omega \subset \mathbb{R}^2$ be a diffeomorphism where Ω is an open, bounded, simply-connected region of the plane. Denote the unit disc by $D^+ \doteq \{z \in \mathbb{C} : |z| < 1\}$, the unit circle by $T \doteq \{z \in \mathbb{C} : |z| = 1\}$, and $D^- \doteq \mathbb{C} \setminus \{D^+ \cup T\}$. We consider $\mathbb{R}^2 \cong \mathbb{C}$ by the standard isomorphism so that $\boldsymbol{\gamma}$ is identified with $\gamma^1(t,s) + i\gamma^2(t,s)$. Then, $(w,\bar{w})$ are independent complex coordinates on Ω where $w \doteq \boldsymbol{\gamma}(t,s)$. Because $\boldsymbol{\gamma}$ is a diffeomorphism, its differential is injective and therefore induces a vector field on Ω via its differential under the rule $(\phi_*X)(f) = X(\phi^*f)$. Consider $\boldsymbol{\gamma}_*\frac{\partial}{\partial t}$. We observe

that this gives a vector field which acts on pushforwards in w of functions on Ω and where the non-degeneracy is ensured by the regularity of the curves $\boldsymbol{\gamma}(t,s)$.

Throughout this article, we will be considering the stationary transport equation $X|_w u(w) = f(w)$, for $w \in \Omega$, $f(w) \in C_c^\infty(\Omega)$ given by the following

$$\mu(w)\frac{\partial u}{\partial w} + \bar{\mu}(w)\frac{\partial u}{\partial \bar{w}} = f(w), \qquad w \in \Omega \tag{13}$$

We will want to use the symmetry of the unit disc which is a priori unavailable to us in this more general domain. There is a unique biholomorphism, $z(\zeta)$, mapping Ω into D^+, the unit disc, with $z(\zeta) = 0$, $z'(\zeta) > 0$ as in [26] and (t,s) give coordinates on D^+ through composition since $\boldsymbol{\gamma}^* z$ maps $\mathbb{R}^2$ into D^+. Because of this equivalence between our initial domain Ω and the unit disc all further results will be presented in the disc. If Ω was all of $\mathbb{R}^2$ (and the Riemann map was consequently unavailable) the method below will still work since $\mathbb{R}^2$ has the needed rotational symmetry.

We therefore use $(z,\bar{z})$ as coordinates on D^+ and have a new vector field on D^+ given by $X|_z = z_* X|_{z(w)}$. and $\mu \to \{z_*\mu\}\frac{\partial z}{\partial w} \circ z^{-1}$ and likewise for $\bar{\mu}$. By slight abuse of notation we denote $\{z_*\mu\}\frac{\partial z}{\partial w} \circ z^{-1}$ by $\mu(z)$ and $\{z_*\bar{\mu}\}\frac{\partial \bar{z}}{\partial \bar{w}} \circ z^{-1}$ by $\bar{\mu}(z)$ so that vector field of interest is

$$X|_z = \mu(z)\frac{\partial}{\partial z} + \bar{\mu}(z)\frac{\partial}{\partial \bar{z}}, \qquad z \in D^+$$

We define $t(z) = z_* w_* t$ and $s(z) = z_* w_* s$, smooth functions on D^+ and suppose that s is real-analytic.

The method of characteristics shows that $X|_z D_1 f(z) = f(z)$ where

$$u(z) = (D_1 f)(z) \doteq \frac{1}{2}\int_{\mathbb{R}} f(z(t_0,s))sign(t(z) - t_0)dt_0 \tag{14}$$

The integral curves of $X|_z$ are just the image of integral curves, i.e. $\boldsymbol{\gamma}^* z^* = (z \circ \boldsymbol{\gamma})^*$. and we define the ray transform of a source function $f(z)$ over the integral curves of $X|_z$ indexed by s to be

$$(If)(s) = \int_{\mathbb{R}} f(z(t,s))dt \tag{15}$$

We will later be using the following extensions of these operators given below:
Symmetrized beam transform

$$(D_\theta \psi)(z) \doteq \frac{1}{2}\int_{\mathbb{R}} \psi(e^{i\theta} z(t_0, s(ze^{-i\theta})))sign(t(ze^{-i\theta}) - t_0)dt_0 \qquad \psi \in L^1(D^+)$$

Ray transform

$$(I\psi)(s, e^{i\theta}) = (I_\theta \psi)(s) \doteq \int_{\mathbb{R}} \psi(e^{i\theta} z(t_0,s))dt_0 \qquad \psi \in L^1(D^+)$$

We will always use θ and $e^{i\theta}$ interchangeably, the meaning determined by context.

We will have occasion to use the **Hilbert transform** H of a function defined as the following Calderón-Zygmund principal value integral operator [36]

$$(H\psi)(x) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{\psi(y)}{x - y} dy \qquad \psi \in L^p(\mathbb{R}), \qquad p > 1 \tag{16}$$

Lastly, we will be using the standard **Poisson kernel** of the unit disc given by $P(z,\theta) = \frac{1-|z|^2}{|1-e^{-i\theta}z|^2}$. We recall that the Poisson kernel generates harmonic solutions

$v(z)$ of the BVP

$$\Delta v = 0, \qquad z \in D^+$$
$$v|_T = g$$

given by $v(z) \doteq \frac{1}{2\pi}\int_T P(z,\theta)g(e^{i\theta})d\theta$; [12, 37] with $g \in C(T)$.

The main purpose of this article will be to show that given suitable conditions on $\mu(z,\bar{z})$ and $s(z,\bar{z})$ that one has an inversion formula for the ray transform given by the following

$$f(z) = \frac{1}{4\pi}\int_0^{2\pi} P(\lambda_i,\theta)X_\theta^\perp H(I_\theta f)(s(ze^{-i\theta}),e^{i\theta}))d\theta \qquad i = 1,...,n$$

where the $\lambda_i(z)$ are functions to be introduced later.

With the above formula established, we present an integrating factor method to estabish a similar reconstruction formula for the attenuated ray transform along the same curves. The above is a type of inversion formula known as a filtered backprojection type [25]. The procedure used to derive the above main result can be best thought of in the following heuristic scheme

1. **Model:** Writing down the linear stationary transport equation for the dynamics
2. **Symmetrizing:** Introducing a rotation parameter $\lambda = e^{i\theta}$ into the integral curves of the transport PDE
3. **Symmetry-Breaking:** Complexifiying the parameter introduced in step 2 by moving λ "off-shell", i.e. $|\lambda| \neq 1$, breaking the rotational symmetry of the problem and generating an elliptic equation
4. **Analysis and Asymptotics:** Evaluating the dependence of solutions to the complexified equation on our parameter λ and examining limiting behavior. These boundary values will be shown to depend on the measured data we are interested in inverting
5. **Reconstruction:** Using holomorphicity of the solutions to write the inversion formulae as Poisson integrals of their asymptotic boundary values found in step 4

The reader may find some benefit from keeping the above rough outline in mind throughout the following. In this section, we have finished step 1. Steps 2 and 3 are handled in the next section. Step 4 is done in the more technical section 4, and the final step is given in section 5.

3. Complexification of the Transport Equation.

We will define the conformal map $\lambda : (z,\bar{z}) \to (\lambda z, \frac{1}{\lambda}\bar{z})$, for $\lambda \in T$ the unit circle. Notice that if $\Phi(\cdot,s)$ is a set of integral curves of D^+, that $z^{-1}(\lambda^*\Phi(\cdot,s))$ are conformally related curves in our original domain Ω. For $\lambda \in \{D^+ \cup D^-\} \setminus \{0,\infty\}$ we consider $\lambda_* X|_z \doteq X_\lambda$ to be the so-called "complexification" of $X|_z$. We remark that $\lambda_* X|_z$ takes the form $\mu(\frac{z}{\lambda},\lambda\bar{z})\lambda\frac{\partial}{\partial z} + \bar{\mu}(\frac{z}{\lambda},\lambda\bar{z})\frac{1}{\lambda}\frac{\partial}{\partial \bar{z}}$ or

$$X_\lambda = \xi(z,\lambda)\frac{\partial}{\partial z} + \rho(z,\lambda)\frac{\partial}{\partial \bar{z}} \qquad \lambda \in D^\pm \setminus \{0,\infty\} \tag{17}$$

with $\frac{1}{\lambda}\xi(z,\lambda) = \mu(z,\lambda) \doteq \lambda_*\mu(z)$ and $\lambda\rho(z,\lambda) = \bar{\mu}(z,\lambda) = \lambda_*\bar{\mu}(z)$. We also define $X_\lambda^\perp = \pm i(-\xi(z,\lambda)\frac{\partial}{\partial z} + \rho(z,\lambda)\frac{\partial}{\partial \bar{z}})$ as a vector field orthogonal to X_λ when $\lambda = e^{i\theta}$. Namely, $X_\theta \cdot X_\theta^\perp = \pm(\xi(z,e^{i\theta}),\rho(z,e^{i\theta}))\cdot(-i\xi(z,e^{i\theta}), i\rho(z,e^{i\theta})) = \pm i(|\xi(z,e^{i\theta})|^2 - |\rho(z,e^{i\theta})|^2) = 0$ in the standard inner product $(\cdot,\cdot) : \mathbb{C}^2 \to \mathbb{C}$. The factor of i is needed to make $X_\theta^\perp u$ real-valued and the choice of $\pm$ is determined by whichever satisfies the condition $X_1^\perp s > 0$. Since $X_1^\perp = a(z) z_* \frac{\partial}{\partial s}$ for some real-valued $a(z)$, this determines $X_1^\perp$ uniquely. We could just as well reparameterize with $-s$ so we will, without any loss of generality, avoid keeping track of signs by just assuming that $X_\lambda^\perp = i(-\xi(z,\lambda)\frac{\partial}{\partial z} + \rho(z,\lambda)\frac{\partial}{\partial \bar{z}})$.

We likewise define $s(z,\lambda)$ and $t(z,\lambda)$ as $\lambda_* s(z)$ and $\lambda_* t(z)$ respectively for $\lambda \in D^\pm \setminus \{0,\infty\}$. A word on notation: $\frac{\partial k}{\partial z}$ and k_z are equivalent, as are $\frac{\partial k}{\partial \bar{z}}$ and $k_{\bar{z}}$, and we will use them interchangably.

We remark that equation (17) *has no direct physical meaning* since the complex parameter λ, when taken to lie away from $T = \partial D^+$, is in some sense artificial and may be best thought of as a complex parameter indexing a class of complex partial differential equations given in (17).

Next we reduce the scope of our consideration to the class of vector fields X_λ consisting only of those satisfying the following **condition H**.

Definition 3.1. A complexified vector field $X_\lambda = \xi(z,\lambda)\frac{\partial}{\partial z} + \rho(z,\lambda)\frac{\partial}{\partial \bar{z}}$, induced in the manner above as $\lambda_* X|_z$, $\lambda \in D^\pm \setminus \{0,\infty\}$ from a real field $X|_z$, is said to be of **type H** if, for each $z \in D^+$, the following conditions hold;

1. $\xi(z,\lambda)$ is holomorphic for $\lambda \in D^+$ and has at least one zero $\lambda_i(z)$ such that $\xi(z,\lambda_i(z)) = 0$
2. $\rho(z,\lambda)$ is a nonvanishing meromorphic function of λ for $\lambda \in D^+$
3. $\frac{\xi(z,\lambda)}{\rho(z,\lambda)}$ is a holomorphic function of λ for $\lambda \in D^+$ and has at least one zero $\lambda = \lambda_i(z) \in D^+$

In addition to the above we will also need a condition on the complexification of the transverse coordinate s, since this will turn out to play an important role in our analysis. For this we define a class of suitable coordinates.

Definition 3.2. Let X_λ be a vector field of **type H** and let $s(z,\lambda) = \lambda_* s(z)$ be the complexification of the real-analytic transverse parameter s indexing the integral curves of X_λ. Then $s(z,\lambda)$ will be called *suitable* whenever the following conditions hold;

1. $s(z,\lambda)$ is a meromorphic function of λ for $\lambda \in D^\pm$ and $\frac{\partial s(z,\lambda)}{\partial z}$, $\frac{\partial s(z,\lambda)}{\partial \bar{z}}$ are meromorphic functions on D^+ away from any singularities of $s(z,\lambda)$
2. For each z there exists an ε such that $s(z,\lambda)$ is analytic in λ for $||\lambda| - 1| \le \varepsilon$
3. $\frac{X_\lambda s(z,\lambda)}{(s(z,\lambda) - s(z_0,\lambda))^2}$ vanishes in the sense of distributions for $z \ne z_0$ at the possible singularities of s, s_z, $s_{\bar{z}}$
4. For $z \ne z_0$, $\frac{s_z(z_0,\lambda)}{s(z,\lambda) - s(z_0,\lambda)}$ is bounded for $\lambda \in D^+ \setminus \{0\}$

We are, in the above, treating z and λ as independent variables. We stress that we are *not* requiring any of the above functions to be holomorphic in the z variable. We pause to present some informal arguments for these definitions.

Some Informal Justifications. While the conditions listed in the first definition above may seem at first unnatural, we remark that holomorphy of $\xi(z,\lambda)$ appears to be the strongest. Aside from the conditions on zeros, or lack thereof, meromorphy itself is weaker and the second and third conditions almost follow from the first.

The definition of suitable s includes conditions that are all related to the behavior of $s(z,\lambda)$, which will be shown to satisfy, aside from isolated critical points, a family of Beltrami equations on D^+ for a Beltrami coefficient depending analytically on λ.

The change of variables induced by λ creates such isolated critical points and these conditions are more or less natural. Indeed, since $X_\theta s(z,\theta) = 0$ and this property survives through complexification almost everywhere, one will have $X_\lambda s(z,\lambda) = 0$ when $(z,\bar{z}) \in S_\lambda = D^+ \setminus \{w : \lim_{w_0 \to w} |s(w_0,\lambda)| = \infty\}$, the complement of the locations of the potential poles of $\lambda_* s$.

Informally, we remark that already the first condition is not too stringent since meromorphy allows for isolated blow-ups. Next, since for each $z \in D^+$, the possible poles occur internally, choosing λ close enough to $T = \partial D^+$ will allow for a local Taylor expansion in λ. For the third of the above we remark that $X_\lambda \lambda_* s = 0$ strongly on S_λ so we only need to worry about the behavior on the complement of S_λ. For this $\lambda_* s \sim \frac{g(z,\lambda)}{(\lambda - p(z))^k}$ with bounded g for $z \to S_\lambda^c$. Then with $|\lambda| < 1$ and for $z \sim w \in S_\lambda^c$ we have that the asymptotic behavior of $\frac{X_\lambda \lambda_* s}{(s(z,\lambda) - s(z_0,\lambda))^2}$ is governed by limiting behavior of

$$(\lambda - p(z))^{2k} \frac{(\xi(z,\lambda) \frac{\partial}{\partial z} \frac{g(z,\lambda)}{(\lambda - p(z))^k} + \rho(z,\lambda) \frac{\partial}{\partial \bar{z}} \frac{g(z,\lambda)}{(\lambda - p(z))^k}}{\pi((g(z,\lambda) - (\lambda - p(z))^k s(z_0,\lambda)))^2}\Bigg|_{z \in S_\lambda^c}$$

and we therefore expect that any distributional singularities occurring from terms like $\frac{\partial}{\partial \bar{z}} \frac{1}{(\lambda - p(z))^k}$ are compensated for by the vanishing of $(\lambda - p(z))^{2k}$. The last condition involves a term like $\frac{s_z(z,\lambda)}{(s(z,\lambda) - s(z_0,\lambda))}$ for $z \neq z_0$. The local behavior of $s(z,\lambda) - s(z_0,\lambda)$ on S_λ is determined by the fact that it solves $X_\lambda s(z,\lambda) = 0$ and vanishes at $z = z_0$. Therefore we have an expansion around z_0 determined by

$$\frac{s_z(z_0,\lambda)}{(s(z,\lambda) - s(z_0,\lambda))} \sim \frac{1}{(z - z_0) - \frac{\xi(z_0,\lambda)}{\rho(z_0,\lambda)}(\bar{z} - \bar{z}_0) + O(\frac{|z - z_0|^{1+\delta}}{s_z(z_0,\lambda)}))} \tag{18}$$

for $\delta > 0$. From this, we see that this the last condition of suitable s is also quite reasonable. The requirements of suitability are in place to prevent pathological examples.

4. **Solving the Complexified Equation.** In trying to solve the complexified transport equation

$$X_\lambda u(z,\lambda) = f(z), \qquad \lambda \in D^+ \setminus \{0\} \tag{19}$$

we will again be changing variables. For this, we will want to collect a few preliminary facts. Because $\frac{\xi(z,\lambda)}{\rho(z,\lambda)}$ is holomorphic in $\lambda \in D^+$ its zeroes are isolated. Also, since $\frac{\xi(z,\lambda)}{\rho(z,\lambda)}$ is holomorphic for $\lambda \in D^+$ and since conformal mappings map boundaries of Jordan domains into boundaries of Jordan domains, then $\frac{\mu(z,e^{i\theta})}{\bar{\mu}(z,e^{i\theta})} = \frac{\mu(y)}{\bar{\mu}(y)}$ for some $y \in T$ and thus $|\frac{\xi(z,\lambda)}{\rho(z,\lambda)}|\Big|_{|\lambda|=1} = 1$. Since we assumed that there is at least one zero λ_i, the maximum principle ensures that $|\frac{\xi(z,\lambda)}{\rho(z,\lambda)}| < 1$ for $\lambda \in D^+$. We then get the following simple lemma.

Lemma 4.1. *$\frac{\xi(z,\lambda)}{\rho(z,\lambda)}$ has a finite number of zeros, $\lambda_i(z)$ with multiplicities $m_i(z)$*

Proof. This is a simple consequence of the argument principle [16]. Namely, one has

$$\sum_i m_i = \frac{1}{2\pi i} \int_{|\lambda|=1} \frac{\frac{\partial}{\partial \lambda} \frac{\xi(z,\lambda)}{\rho(z,\lambda)}}{\xi(z,\lambda)} \rho(z,\lambda) d\lambda$$

and $\frac{\xi(z,\lambda)}{\rho(z,\lambda)}$ is holomorphic, hence so is $\frac{\partial}{\partial\lambda}\frac{\xi(z,\lambda)}{\rho(z,\lambda)}$, on the region D^+. They are also both continuous on T. Therefore, $|\frac{\partial}{\partial\lambda}\frac{\xi(z,\lambda)}{\rho(z,\lambda)}| < M < \infty$ for $\lambda \in \overline{D^+}$. Thus,

$$\sum_i m_i \leq \frac{1}{2\pi}|\int_{|\lambda|=1} \frac{\frac{\partial}{\partial\lambda}\frac{\xi(z,\lambda)}{\rho(z,\lambda)}}{\xi(z,\lambda)}\rho(z,\lambda)d\lambda| < \frac{1}{2\pi}\int_0^{2\pi} Md\theta = M$$

□

Henceforth λ_i will always be used to indicate a value in the unit disc for which $\frac{\xi(z,\lambda)}{\rho(z,\lambda)}$ (and ξ) vanishes. The bounded holomorphic functions mapping the unit disc onto itself and having a finite number of zeroes can be uniquely written as a finite Blashke product so that $\frac{\xi(z,\lambda)}{\rho(z,\lambda)}$ can be given in the form $\zeta(z)\Pi_{i=1}^n(\frac{\lambda-\lambda_i}{1-\lambda\bar{\lambda}_i})^{m_i}$ with $|\zeta(z)| = 1$, and with m_i and λ_i possibly depending on z; see [10, 15].

Furthermore, since $|\frac{\xi(z,\lambda)}{\rho(z,\lambda)}| < 1$ for $\lambda \in D^+$ we also have that the complexified transport equation $X_\lambda u(z,\lambda) = f(z)$ can be rewritten as

$$u_{\bar{z}}(z,\lambda) = \frac{\xi(z,\lambda)}{\rho(z,\lambda)}u_z(z,\lambda) + \frac{f(z)}{\rho(z,\lambda)} \qquad \lambda \in D^+ \setminus \{0\} \tag{20}$$

which is a holomorphically forced Beltrami equation, [3, 8, 5]. If $u_z(z,0)$ is bounded this will hold at $\lambda = 0$ as well.

Letting $S_\lambda = D^+ \setminus \{w : \lim_{w_0\to w}|s(w_0,\lambda)| = \infty\}$ denote the complement of the locations of the potential poles of $\lambda_* s$ we see that for $\lambda \in D^+ \setminus \{0\}$, $X_\lambda s(z,\lambda) = 0$ on that region so that s is still a constant of the dynamics. This is obvious from the fact that integral curves are mapped by diffeomorphisms to integral curves, however to be precise, when $|\lambda| \neq 0$,

$$\begin{aligned} X_\lambda s(z,\lambda) &= \lambda_* X|_z \lambda_* s(z) = \lambda_* z_* w_* \frac{\partial}{\partial t}\lambda_* z_* w_* s = (\lambda\circ z\circ w)_* \frac{\partial}{\partial t}(\lambda\circ z\circ w)_* s \\ &= (\lambda\circ z\circ w)_* \frac{\partial s}{\partial t} = 0 \end{aligned}$$

since s and t are independent coordinates. Thus for $\lambda \neq 0$,

$$\xi(z,\lambda)\frac{\partial s(z,\lambda)}{\partial z} + \rho(z,\lambda)\frac{\partial s(z,\lambda)}{\partial \bar{z}} = 0, \qquad (z,\bar{z}) \in S_\lambda$$

The consideration of S_λ is an unfortunate artifact of having brought in potential singularities at $|z| = 1$ to the interior upon complexification. We will need the following result to set up a fundamental equation in s.

Lemma 4.2. *On $0 < |\lambda| < 1$ the Jacobian $J_s(z,\lambda) \doteq |s_z(z,\lambda)|^2 - |s_{\bar{z}}(z,\lambda)|^2$ is positive*

Proof. Since $(t,s) \mapsto (z,\bar{z})$ is a diffeomorphism and $\lambda : (z,\bar{z}) \mapsto (\frac{z}{\lambda}, \bar{z}\lambda)$ is conformal on $0 < |\lambda| < 1$

$$\begin{vmatrix} \frac{\partial s(z,\lambda)}{\partial(\frac{z}{\lambda})} & \frac{\partial t(z,\lambda)}{\partial(\frac{z}{\lambda})} \\ \frac{\partial s(z,\lambda)}{\partial(\bar{z}\lambda)} & \frac{\partial t(z,\lambda)}{\partial(\bar{z}\lambda)} \end{vmatrix} \neq 0 \tag{21}$$

so that

$$|\frac{\partial s(z,\lambda)}{\partial z}\frac{\partial t(z,\lambda)}{\partial \bar{z}} - \frac{\partial s(z,\lambda)}{\partial \bar{z}}\frac{\partial t(z,\lambda)}{\partial z}| \leq \frac{2|s_z(z,\lambda)|}{|\lambda||\rho(z,\lambda)|}$$

implies $|s_z(z,\lambda)|^2 \neq 0$. Then,

$$\partial s(z) = |s_z(z,\lambda)|^2 - |\frac{\xi(z,\lambda)}{\rho(z,\lambda)} s_z(z,\lambda)|^2 \geq |s_z(z,\lambda)|^2 (1 - |\frac{\xi(z,\lambda)}{\rho(z,\lambda)}|^2) > 0$$

since $|\frac{\xi(z,\lambda)}{\rho(z,\lambda)}| < 1$ for $\lambda \in D^+$. □

Fixing λ, we work on $(z,\bar{z}) \in S_\lambda$. Since $X_\lambda s(z,\lambda) = 0$, $s_* X_\lambda = s_* X_\lambda \bar{s}(z,\lambda) \frac{\partial}{\partial \bar{s}}$. We are interested in solving $X_\lambda G_\lambda(z;z_0) = \delta(z - z_0)$ and we can achieve this by solving $s_* X_\lambda \bar{s}(z,\lambda) \frac{\partial}{\partial \bar{s}} (s_* G_\lambda) = |J_s(z,\lambda)| \delta(s(z,\lambda) - s_0)$. For this we will need to compute the term $s_* X_\lambda \bar{s}(z,\lambda)$. Observe that

$$\xi(z,\lambda) = -\rho(z,\lambda) \frac{\frac{\partial s(z,\lambda)}{\partial \bar{z}}}{\frac{\partial s(z,\lambda)}{\partial z}}$$

whence

$$\begin{aligned}
\xi(z,\lambda)\frac{\partial \bar{s}(z,\lambda)}{\partial z} + \rho(z,\lambda)\frac{\partial \bar{s}(z,\lambda)}{\partial \bar{z}} &= -\rho(z,\lambda) \frac{\frac{\partial s(z,\lambda)}{\partial \bar{z}}}{\frac{\partial s(z,\lambda)}{\partial z}} \frac{\partial \bar{s}(z,\lambda)}{\partial z} + \rho(z,\lambda)\frac{\partial \bar{s}(z,\lambda)}{\partial \bar{z}} \\
&= \frac{\rho(z,\lambda)}{\frac{\partial s(z,\lambda)}{\partial z}} (|s_z(z,\lambda)|^2 - |s_{\bar{z}}(z,\lambda)|^2) \\
&= \frac{1}{Q(z,\lambda)} J_s(z,\lambda)
\end{aligned}$$

with $Q(z,\lambda) \doteq \frac{\frac{\partial s(z,\lambda)}{\partial z}}{\rho(z,\lambda)}$. By recalling that $|\frac{\xi}{\rho}| > 1$ for $|\lambda| > 1$ and going through the preceding lemma *mutatis mutandis* we see that $J_s(z,\lambda)$ is likewise negative on D^- and hence the Jacobian of $s(z,\lambda)$ *switches sign* when $\lambda \in D^\pm$ so that, generally, our fundamental equation may be written compactly as follows

$$s_* \frac{1}{Q(z,\lambda)} \frac{\partial}{\partial \bar{s}} s_* G_\lambda = sign(1 - |\lambda|) \delta(s(z,\lambda) - s(z_0,\lambda))$$

Using $\frac{\partial}{\partial z} \frac{1}{\pi \bar{z}} = \delta(z)$ as shown in [16] see see this equals $G_\lambda(z;z_0) = \frac{sgn(1-|\lambda|)Q(z_0,\lambda)}{\pi(s(z) - s(z_0))}$. Then, for $(z,\bar{z}) \in S_\lambda$ we have

$$G_\lambda(z;z_0) = \frac{sgn(1-|\lambda|)\frac{1}{\rho(z_0,\lambda)}\frac{\partial s(z,\lambda)}{\partial z}|_{z_0}}{\pi(s(z,\lambda) - s(z_0,\lambda))}, \qquad \lambda \in D^\pm \setminus \{0,\infty\}$$

Remark 1. We will only make use of results in our formula which follow from **condition H** and thus results like the above are only used when $\lambda \in D^+$. We will however present many results for $\lambda \in D^-$ with the understanding that given an appropriate generalization of **condition H** (involving constraints on ξ and ρ for $\lambda \in \mathbb{C} \setminus \bar{D}^+$) and the related suitablility of s, the results are true. The advantage to this approach is it makes apparent the symmetries and parallels of several of the formulae for $\lambda \in D^\pm$. Thus, in the "$-$" versions of several results, **condition H** would need to be augmented appropriately.

Next, we show that $G_\lambda(z;z_0)$ defined in equation (22) extends to satisfy $X_\lambda G_\lambda(z;z_0) = \delta(z - z_0)$ on $z \in D^+$. For this we note that $X_\lambda G_\lambda(z;z_0)$ behaves weakly, in a vicinity of S_λ, as

$$\lim_{z \to w \in D^+ \setminus S_\lambda} \frac{sgn(1-|\lambda|)\frac{1}{\rho(z_0,\lambda)}\frac{\partial s(z,\lambda)}{\partial z}|_{z_0} X_\lambda s(z,\lambda)}{(s(z,\lambda) - s(z_0,\lambda))^2}$$

which, by assumption vanishes when integrated against a smooth function. The fundamental equation, therefore, is satisfied for all $z \in D^+$, i.e.

$$G_\lambda(z; z_0) = \frac{sgn(1-|\lambda|)\frac{1}{\rho(z_0,\lambda)}\frac{\partial s(z,\lambda)}{\partial z}|_{z_0}}{\pi(s(z,\lambda)-s(z_0,\lambda))}, \quad \lambda \in D^\pm \setminus \{0,\infty\}, \quad z \in D^+ \tag{22}$$

Due to the meromorphy of the terms appearing in $G_\lambda(z; z_0)$, together with the boundedness of $\frac{s_z(z_0,\lambda)}{s(z,\lambda)-s(z_0,\lambda)}$ after integration against a smooth enough function $u(z,\lambda) = \int_{\mathbb{C}} G_\lambda(z; z_0) f(z_0) d\mu(z_0)$ remains bounded for $\lambda \in D^+ \setminus \{0\}$ since the preceding arguments hold also when λ approaches singularities of $s(z,\lambda)$ and $s_z(z,\lambda)$. The expansion $s(z,\lambda) \sim c_0 + \frac{c_1 z}{\lambda} + c_2 \bar{z}\lambda + O((z\frac{1}{\lambda} + \bar{z}\lambda)^2)$ near zero shows that s and s_z have the same order of possible vanishing or blow-up at $\lambda \to 0$. Therefore, $G_\lambda(z; z_0)$ is analytically extendible to $\lambda = 0$ and we obtain an analytic extension of the solution $u(z,\lambda)$, which we also denote u. We have thereby established the following proposition.

Proposition 1. *For each $z \in D^+$, the solution $u(z,\lambda)$ satisfies $\partial_{\bar{\lambda}} u(z,\lambda) = 0$ for $\lambda \in D^\pm$*

By using $\frac{\partial}{\partial z} G_\lambda(z; z_0) = G_\lambda(z; z_0) \frac{s_z(z,\lambda)}{s(z,\lambda)-s(z_0,\lambda)}$ and performing similar estimates as above one obtains that, for every $z \in D^+$, $u_z(z,\lambda)$ is complex-analytic in $\lambda \in D^\pm$. The corresponding result for $u_{\bar{z}}(z,\lambda)$ follows from the holomorphy of the right hand side of $u_{\bar{z}}(z,\lambda) = -\frac{\xi(z,\lambda)}{\rho(z,\lambda)} u_z(z,\lambda) + \frac{f(z)}{\rho(z,\lambda)}$.

Boundary Behavior. We will be using the boundary values $u(z,\lambda)|_{\lambda \in T}$ to arrive at a reconstruction formula. For this we notice that

$$\begin{aligned}
\frac{\partial(t,s)}{\partial(z,\bar{z})} \frac{\frac{\partial \bar{z}}{\partial t}}{\frac{\partial s}{\partial z}} &= \frac{\partial \bar{z}}{\partial t}\frac{\partial s}{\partial \bar{z}}\frac{\partial t}{\partial z}\frac{1}{\frac{\partial s}{\partial z}} - \frac{\partial \bar{z}}{\partial t}\frac{\partial t}{\partial \bar{z}} \\
&= -(\frac{\partial z}{\partial t}\frac{\partial t}{\partial z} + \frac{\partial \bar{z}}{\partial t}\frac{\partial t}{\partial \bar{z}}) \\
&= -z_* \frac{\partial t}{\partial t}
\end{aligned}$$

and therefore we may rewrite $G_\lambda(z; z_0)$ appearing in equation (22) as follows

$$G_\lambda(z, z_0) = -\lambda_* \frac{\frac{\partial(t,s)}{\partial(z,\bar{z})}\Big|_{z_0}}{\pi(s(z)-s(z_0))} \tag{23}$$

We may now prove the following consequence of this fact.

Proposition 2. *The non-tangential limits $u_\pm(z, e^{i\theta}) \doteq \lim_{D^\pm \ni \lambda \to e^{i\theta}} u(z,\lambda)$ are given by the following*

$$u_\pm(z, e^{i\theta}) = \mp \frac{1}{2i}(HI_\theta f)(s(e^{-i\theta}z), \theta) + (D_\theta f)(z)$$

where, in the above, the Hilbert transform H is taken with respect to the first variable.

Proof. Let $\psi \in C_c^\infty(D^+)$. First we examine $\frac{1}{s(z,\lambda)-s(z_0,\lambda)}$ when $\lambda = 1-\varepsilon$ ($\varepsilon << 1$). With $'$ denoting differentiation in λ we use the fact that $s(z, 1-\varepsilon) = s(z,1) -$

$\varepsilon s'(z,1) + o(\varepsilon^2)$ together with $X_\lambda s(z,\lambda) = 0$ for λ near enough to ∂D^+ to get

$$O(1): \qquad (\xi(z,1)\frac{\partial}{\partial z} + \rho(z,1)\frac{\partial}{\partial \bar{z}})s(z,1) = 0$$

$$O()_\varepsilon: \qquad (\xi(z,1)\frac{\partial}{\partial z} + \rho(z,1)\frac{\partial}{\partial \bar{z}})s'(z,1) = -(\xi'(z,1)\frac{\partial}{\partial z} + \rho'(z,1)\frac{\partial}{\partial \bar{z}})s(z,1)$$

and

$$\begin{aligned} -(\xi'(z,1)\frac{\partial}{\partial z} + \rho'(z,1)\frac{\partial}{\partial \bar{z}})s(z,1) &= -(\xi'(z,1) - \rho'(z,1)\frac{\xi(z,1)}{\rho(z,1)})s_z(z,1) \\ &= -\xi(z,1)s_z(z,1)\{\frac{\xi'(z,1)}{\xi(z,1)} - \frac{\rho'(z,1)}{\rho(z,1)}\} \\ &= -\xi(z,1)s_z(z,1)\frac{(\frac{\partial}{\partial\lambda}\frac{\xi}{\rho})\Big|_{\lambda=1}}{\frac{\xi(z,1)}{\rho(z,1)}} \end{aligned} \tag{24}$$

so that $X_1 is'(z,1) = -i\xi(z,1)s_z(z,1)\frac{(\frac{\partial}{\partial\lambda}\frac{\xi}{\rho})\big|_{\lambda=1}}{\frac{\xi(z,1)}{\rho(z,1)}}$. By a similar argument one can show that $X_1 is'(z,1) = i\rho(z,1)s_{\bar{z}}(z,1)\frac{(\frac{\partial}{\partial\lambda}\frac{\xi}{\rho})\big|_{\lambda=1}}{\frac{\xi(z,1)}{\rho(z,1)}}$ whereby we see that

$$X_1 is'(z,1) = \frac{1}{2}\frac{(\frac{\partial}{\partial\lambda}\frac{\xi}{\rho})\Big|_{\lambda=1}}{\frac{\xi(z,1)}{\rho(z,1)}} X_1^\perp s(z,1) \tag{25}$$

Since $\frac{\xi}{\rho}$ is given as a finite Blashke product $\zeta(z)\Pi_{i=1}^n(\frac{\lambda-\lambda_i(z)}{1-\lambda\bar{\lambda}_i})^{m_i(z)}$, we see that $\frac{\frac{\partial}{\partial\lambda}\frac{\xi(z,\lambda)}{\rho(z,\lambda)}}{\frac{\xi(z,\lambda)}{\rho(z,\lambda)}} = \sum_{j>0} m_j \frac{1-|\lambda_j|^2}{(\lambda-\lambda_j)(1-\bar{\lambda}_j\lambda)}$ so that $\frac{(\frac{\partial}{\partial\lambda}\frac{\xi}{\rho})\big|_{\lambda=1}}{\frac{\xi(z,1)}{\rho(z,1)}} > 0$, which, when combined with $X_1^\perp s(z,1) > 0$ gives from (25) that $X_1 is'(z,1) > 0$. This implies that

$$sgn(is'(z,1) - is'(z_0,1)) = sgn(t(z,1) - t(z_0,1))$$

Testing against a compactly supported $\psi(z_0)$ ensures that we may apply a similar Taylor expansion for $s(z_0,\lambda)$ in integration in the z_0 variable. Therefore we use the distributional Plemelj relation $\frac{1}{ix+\varepsilon} \xrightarrow{|\varepsilon|\searrow 0} \frac{1}{ix} + sgn(\varepsilon)\pi\delta(x)$ to see that, distributionally, one has that $\frac{1}{s(z,\lambda)-s(z_0,\lambda)}$ tends, as $\lambda \to 1$, to

$$\frac{1}{s(z,1) - s(z_0,1)} - isgn(is'(z,1) - is'(z_0,1))\delta(s(z,1) - s(z_0,1))$$

Since $\lim_{\lambda\to 1}\lambda_*\frac{\partial(t,s)}{\partial(z,\bar{z})} = \frac{\partial(t,s)}{\partial(z,\bar{z})}$ we see that the preceding considerations allow us to conclude that the following holds

$$\begin{aligned} u_+(z,\theta) &\doteq \lim_{\varepsilon\searrow 0}\int_{D^+} G_{1-\varepsilon}(z;z_0)\psi(z_0)d\mu(z_0) \\ &= \frac{1}{2\pi i}\int_{D^+}\frac{\psi(z_0)\ \frac{\partial(t,s)}{\partial(z,\bar{z})}\Big|_{z_0}}{s(z,1) - s(z_0,1)}dz_0 \wedge d\bar{z}_0 \\ &\quad - \frac{1}{2}\int_{D^+}\psi(z_0)sgn(is'(z,1) - is'(z_0,1))\delta(s(z,1) - s(z_0,1))\ \frac{\partial(t,s)}{\partial(z,\bar{z})}\Big|_{z_0} dz_0 \wedge d\bar{z}_0 \end{aligned}$$

Then, with $\kappa = \pm 1$ determined by the orientation of the Jacobian of $(t,s) \to (z,\bar{z})$ we have that the above equals

$$\kappa\{\frac{1}{2\pi i}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{\psi(z(t_0,s_0))dt_0ds_0}{s(z,1)-s_0}-\frac{1}{2}\int_{\mathbb{R}}\psi(z(t_0,s_0))sgn(t(z,1)-t(z_0,1)dt_0\}$$

Since, by continuity on the limit, $\Re u_+(z,\theta)$ is determined to be f and not $-f$, we have that $\kappa=-1$ and get the following

$$u_+(z,1)=-\frac{1}{2i}H(I_\theta\psi)(s(z),1)+(D_1\psi)(z)$$

One can see this also since $i|\mu|^2\frac{\partial(t,s)}{\partial(z,\bar{z})}=X_1^\perp s(z)>0$ and therefore $\frac{\partial(t,s)}{\partial(z,\bar{z})}$ is $-ib(z)$ for $b(z)$ positive real-valued.

For the general case, $\psi(z_0e^{i\theta})G_{e^{i\theta}}(z;z_0)=\psi(z_0e^{i\theta})G_1(e^{-i\theta}z;e^{-i\theta}z_0)$ together with invariance of the measure under the complexification map; i.e. $(e^{i\theta})^*d\mu(z_0)=d\mu(z_0)$ shows that we get the following boundary values for general $\theta\in[0,2\pi)$

$$u_+(z,e^{i\theta})=-\frac{1}{2i}H(I_\theta\psi)(s(ze^{-i\theta}),e^{i\theta})+(D_\theta\psi)(z). \tag{26}$$

An identical argument for $u_-(z,e^{i\theta})$ shows that

$$u_\pm(z,e^{i\theta})=\mp\frac{1}{2i}H(I_\theta\psi)(s(ze^{-i\theta}),e^{i\theta})+(D_\theta\psi)(z)$$

□

Incidentally, from the above arguments we see that

$$H(I_\theta f)(s(z,e^{-i\theta}),\theta)\in\ker X_\theta$$

Later, in using an integrating factor approach to solve the attenuated ray transform, we will see that this property will be to our benefit. In fact, constancy of the Hilbert transform of the ray transform of the attenuation coefficient is a pervasive feature of analytical inversion formula for the attenuated ray transform; see for instance [35].

5. **Inversion Formulae.**

5.1. **No Attenuation.** We can now prove our main result.

Theorem 5.1. *Let $X_\lambda=\xi(z,\lambda)\frac{\partial}{\partial z}+\rho(z,\lambda)\frac{\partial}{\partial\bar{z}}$ be a vector field of **type H**, $s(z,\lambda)$ be suitable, $\xi(z,\lambda_i(z))=0$ for $i=1,...,n$ and $f(z)\in C_c^\infty(D^+)$. Then*

$$f(z)=\frac{1}{4\pi}\int_0^{2\pi}P(\lambda_i,\theta)X_\theta^\perp H(I_\theta f)(s(ze^{-i\theta}),e^{i\theta})d\theta$$

gives an exact reconstruction formula for the density f ba sed on the data $I_\theta f$ of ray transforms of f over the integral curves of X_θ.

Proof. The real and imaginary parts of complex-analytic functions are harmonic, so with $P(z,\theta)$ the Poisson kernel of the unit disc one has, on using the boundary values given (26) given in Proposition 2, that

$$\begin{aligned}X_{\lambda_i}u(z,\lambda_i)=&\frac{i}{4\pi}\int_0^{2\pi}P(\lambda_i,\theta)X_\theta H(I_\theta f)(s(ze^{-i\theta}),e^{i\theta})d\theta\\&+\frac{1}{2\pi}\int_0^{2\pi}P(\lambda_i,\theta)X_\theta(D_\theta f)(z)d\theta\end{aligned} \tag{27}$$

so that

$$X_{\lambda_i} u(z, \lambda_i) = f(z) + \frac{i}{4\pi} \int_0^{2\pi} P(\lambda_i, \theta) X_\theta H(I_\theta f)(s(ze^{-i\theta}), e^{i\theta}) d\theta$$

whereas

$$X_{\lambda_i}^{\perp} u(z, \lambda_i) = \frac{i}{4\pi} \int_0^{2\pi} P(\lambda_i, \theta) X_\theta^{\perp} H(I_\theta f)(s(ze^{-i\theta}), e^{i\theta}) d\theta$$

$$+ \frac{1}{2\pi} \int_0^{2\pi} P(\lambda_i, \theta) X_\theta^{\perp} (D_\theta f)(z) d\theta \tag{28}$$

Then since $X_\lambda = \xi(z, \lambda) \frac{\partial}{\partial z} + \rho(z, \lambda) \frac{\partial}{\partial \bar{z}}$, $X_\lambda^{\perp} = i(-\xi(z, \lambda) \frac{\partial}{\partial z} + \rho(z, \lambda) \frac{\partial}{\partial \bar{z}})$ and $\xi(z, \lambda_i) = 0$, we have that

$$i X_{\lambda_i} u(z, \lambda_i) = X_{\lambda_i}^{\perp} u(z, \lambda_i)$$

so that, on equating real and imaginary parts of (27) and (28), we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} P(\lambda_i, \theta) X_\theta^{\perp} (D_\theta f)(z) d\theta = -\frac{1}{4\pi} \int_0^{2\pi} P(\lambda_i, \theta) X_\theta H(I_\theta f)(s(ze^{-i\theta}), e^{i\theta}) d\theta$$

and

$$f(z) = \frac{1}{4\pi} \int_0^{2\pi} P(\lambda_i, \theta) X_\theta^{\perp} H(I_\theta f)(s(ze^{-i\theta}), e^{i\theta}) d\theta \tag{29}$$

□

Taking into account the presence of the signum function in the Green's function (23) it's clear that formula (29) could just as well be written in terms of the jump function (from the viewpoint of $D^{\pm}$)

$$\phi(z, e^{i\theta}) \doteq u_+(z, e^{i\theta}) - u_-(z, e^{i\theta}) = iH(I_\theta f)(s(ze^{-i\theta}), e^{i\theta})$$

as

$$f(z) = \frac{1}{4\pi} \int_0^{2\pi} P(\lambda_i, \theta) X_\theta^{\perp} (-i\phi(z, e^{i\theta})) d\theta$$

an observation which will be notationally useful in the next section. Recalling our previous **remark** about using only results from D^+ we could just as well use

$$\phi(z, e^{i\theta}) \doteq 2i\Im(u_+(z, e^{i\theta}))$$

and remember that invoking D^- is only a useful mnemonic.

5.2. **Attenuated Ray Transform and Inversion Formulae.** We add a positive and real-valued attenuation term $a(z) \in C_c^\infty(D^+)$ to the complexified stationary transport equation to get

$$(X_\lambda + a(z)) u(z, \lambda) = f(z) \qquad \lambda \in D^{\pm} \setminus \{0, \infty\}$$

Using our Green's function $G_\lambda(z; z_0)$ determined in (22) we define

$$h(z, \lambda) \doteq \int_{D^+} G_\lambda(z; z_0) a(z_0) d\mu(z_0)$$

In exactly the same manner as in the previous sections we may extend the above to a formula which holds on $(z, \lambda) \in D^+ \times D^+$.

We will be using an integrating factor approach as follows

$$e^{h(z,\lambda)} X_\lambda u(z, \lambda) + e^{h(z,\lambda)} a(z) u(z, \lambda) = e^{h(z,\lambda)} f(z)$$

so that

$$X_\lambda e^{h(z,\lambda)} u(z,\lambda) = e^{h(z,\lambda)} f(z)$$

whence

$$u(z,\lambda) = \int_{D^+} G_\lambda(z;z_0) e^{h(z_0,\lambda)-h(z,\lambda)} f(z_0)\, d\mu(z_0) \tag{30}$$

The Green's function, as discussed previously, is analytic for $z \neq z_0$, so that holomorphy of $e^{h(z_0,\lambda)}$ is retained after integration in z_0. Therefore equation (30) defines a complex-analytic function in λ, for each z extendible to hold on all $\lambda \in D^+$.

Now, by Proposition 2 one has

$$h_\pm(z,e^{i\theta}) = \mp \frac{1}{2i}(HI_\theta a)(s(ze^{-i\theta}),\theta)) + (D_\theta a)(z)$$

Therefore, in another application of Proposition 2 we have the solution of the attenuated transport equation admits the following nontangential boundary values as $|\lambda| \to 1^\mp$

$$\begin{aligned} u_\pm(z,e^{i\theta}) &= \frac{\mp e^{-h_\pm(z,e^{i\theta})}}{2i}[HI_\theta\{e^{h_\pm(\cdot,e^{i\theta})}f\}(s(ze^{-i\theta}),\theta) \mp 2i(D_\theta e^{h_\pm(\cdot,e^{i\theta})}f)(z)] \\ &= \frac{\mp e^{-h_\pm(z,e^{i\theta})}}{2i}[HI_\theta\{e^{\frac{\mp 1}{2i}(HI_\theta)a(s(e^{-i\theta}\cdot),\theta)}f(\cdot)e^{(D_\theta a)(\cdot)}\}(s(ze^{-i\theta}),\theta) \\ &\mp\ 2i(D_\theta e^{\frac{\mp 1}{2i}(HI_\theta)a(s(e^{-i\theta}\cdot),\theta)}f(\cdot)e^{(D_\theta a)(\cdot)})(z)] \end{aligned}$$

Define the attenuated ray transform as

$$(I_{a,\theta}f)(s) \doteq I_\theta(f(\cdot)e^{(D_\theta a)(\cdot)})(s)$$

and recall that I_θ involves integration in t, not s (as does D_θ) and therefore since $HI_\theta a(s(z,e^{-i\theta}),\theta) \in \ker X_\theta$, $HI_\theta a(s(z,e^{-i\theta}),\theta)$ is constant on the curves of integration in t and therefore may be pulled through the $I_{a,\theta}$ integrals as

$$\begin{aligned} u_\pm(z,e^{i\theta}) = \frac{\mp e^{-h_\pm(z,e^{i\theta})}}{2i} H(e^{\frac{\mp 1}{2i}(H(I_\theta a)(s(e^{-i\theta}\cdot),\theta)}I_{a,\theta}f)(s(ze^{-i\theta}),\theta) \\ + e^{-(D_\theta a)(z)}(D_\theta f(\cdot)e^{(D_\theta a)(\cdot)})(z) \end{aligned}$$

Therefore, the difference in the above limits yields the following

$$\begin{aligned} \phi(z,e^{i\theta}) &\doteq (u_+ - u_-)(z,e^{i\theta}) \\ &= -\frac{e^{-h_-(z,e^{i\theta})}}{2i} H(e^{\frac{1}{2i}H(I_\theta a)(s(e^{-i\theta}\cdot),\theta)}I_{a,\theta}f)(s(ze^{-i\theta}),\theta) \\ &\quad - \frac{e^{-h_+(z,e^{i\theta})}}{2i} H(e^{-\frac{1}{2i}H(I_\theta a)(s(e^{-i\theta}\cdot),\theta)}I_{a,\theta}f)(s(ze^{-i\theta}),\theta) \\ &= -\frac{e^{-(D_\theta a)(z)}}{2i}\{e^{\frac{1}{2i}H(I_\theta a)(s(ze^{-i\theta}),\theta)} H(e^{\frac{1}{2i}H(I_\theta a)(s(e^{-i\theta}\cdot),\theta)}I_{a,\theta}f) \\ &\quad + e^{-\frac{1}{2i}H(I_\theta a)(s(ze^{-i\theta}),\theta)} H(e^{-\frac{1}{2i}H(I_\theta a)(s(e^{-i\theta}\cdot),\theta)}I_{a,\theta}f)\}(s(ze^{-i\theta}),\theta) \end{aligned}$$

To simplify this expression, we define the following filtered Hilbert transform

$$H_a : g \mapsto \{CH(Cg)\}(s(ze^{-i\theta}),\theta) + \{SH(Sg)\}(s(ze^{-i\theta}),\theta) \tag{31}$$

for $g(s,\theta) \in C^\infty(\mathbb{R}\times T)$, with

$$C \doteq \cos(\frac{H(I_\theta a)(s(ze^{-i\theta}),\theta)}{2}), \qquad S \doteq \sin(\frac{H(I_\theta a)(s(ze^{-i\theta}),\theta)}{2})$$

Then

$$\begin{aligned}\phi(z,e^{i\theta}) &= -\frac{e^{-(D_\theta a)(z)}}{2i}[(C-iS)H\{(C-iS)I_{a,\theta}f\} \\ &\quad + (C+iS)H\{(C+iS)I_{a,\theta}f\}](s(ze^{-i\theta}),\theta) \\ &= ie^{-(D_\theta a)(z)}\Re\{(C-iS)H[(C-iS)I_{a,\theta}f](s(ze^{-i\theta}),\theta)\} \\ &= ie^{-(D_\theta a)(z)}(CH(CI_{a,\theta}f)(s(ze^{-i\theta}),\theta) + SH(SI_{a,\theta}f)(s(ze^{-i\theta}),\theta)) \\ &\doteq ie^{-(D_\theta a)(z)}(H_a I_{a,\theta}f)(s(ze^{-i\theta}),\theta) \qquad (32)\end{aligned}$$

We then can proceed in a manner similar to before since we have that $e^{h(z,\lambda)}u(z,\lambda)$ (along with its derivatives) is holomorphic and solves $X_\lambda e^{h(z,\lambda)}u(z,\lambda) = e^{h(z,\lambda)}f(z)$.

We have the following theorem on the attenuated ray transform.

Theorem 5.2. *Let $X_\lambda = \xi(z,\lambda)\frac{\partial}{\partial z} + \rho(z,\lambda)\frac{\partial}{\partial \bar z}$ be a vector field of **type H**, $s(z,\lambda)$ be suitable, $\xi(z,\lambda_i(z)) = 0$ for $i = 1,...,n$, and $a(z), f(z) \in C_c^\infty(D^+)$. Let $u(z,\lambda)$ solve $X_\lambda u(z,\lambda) + a(z)u(z,\lambda) = f(z)$ on $(z,\lambda) \in D^+\times D^+$ and define H_a by expression (31) and $p(z) \doteq \Re\{u(z,\lambda_i(z))\}$. Then*

$$f(z) - a(z)p(z) = \frac{1}{4\pi}\int_0^{2\pi} P(\lambda_i,\theta)X_\theta^\perp(e^{-(D_\theta a)(z)}H_a I_{a,\theta}f)(s(ze^{-i\theta}),\theta))d\theta$$

gives an exact reconstruction formula for the density f based on the data $I_{a,\theta}f$ of attenuated ray transforms of f over the integral curves of X_θ modulo the values taken on by the solution $u(z,\lambda_i(z))$.

Proof. The proof proceeds as in the proof of Theorem 5.1. Since $X_\lambda u(z,\lambda) + a(z)u(z,\lambda) = f(z)$, and $\lim_{\lambda\to\lambda_i} X_\lambda u(z,\lambda) = \lim_{\lambda\to\lambda_i} -iX_\lambda^\perp u(z,\lambda)$ we verify that

$$\lim_{\lambda\to\lambda_i}\{-iX_\lambda^\perp u(z,\lambda) + a(z)u(z,\lambda)\} = f(z)$$

whereby

$$f(z) = a(z)\Re\{a(z)u(z,\lambda_i(z))\} + \frac{1}{4\pi}\int_0^{2\pi} P(z,\lambda_i(z))X_\theta^\perp(-i\phi(z,\theta))d\theta$$

The result follows from (32).

□

A comparison of equations (18) and (22) for the Green's function of X_λ shows that we expect the behavior of the solution $u(z,\lambda)$ to behave locally as

$$\frac{e^{h(z_0,\lambda)-h(z,\lambda)}f(z_0)}{\rho(z_0,\lambda)(z-z_0) - \xi(z_0,\lambda)(\bar z - \bar z_0) + O(\frac{|z-z_0|^{1+\delta}}{s_z(z_0,\lambda)})}, \qquad \delta > 0, \quad z \neq z_0$$

Therefore, if $\rho(z_0,\lambda)$ has a singularity at $\lambda = \lambda_i(z)$ our solution will vanish there. Equivalently, if $\lim_{\lambda\to\lambda_i}|s(z,\lambda)| = \infty$ then $u(z,\lambda_i(z)) = 0$. In this case we have the following simple corollary of the preceding.

Corollary 1. *Let $X_\lambda = \xi(z,\lambda)\frac{\partial}{\partial z} + \rho(z,\lambda)\frac{\partial}{\partial \bar z}$ be a vector field of **type H**, $s(z,\lambda)$ be suitable, $\xi(z,\lambda_i(z)) = 0$ for $i = 1,...,n$, and $a(z), f(z) \in C_c^\infty(D^+)$. Let $u(z,\lambda)$*

solve $X_\lambda u(z,\lambda) + a(z)u(z,\lambda) = f(z)$ *on* $(z,\lambda) \in D^+ \times D^+$*, define* H_a *by expression (31) and suppose that* $u(z,\lambda_i(z)) = 0$*. Then*

$$f(z) = \frac{1}{4\pi}\int_0^{2\pi} P(\lambda_i,\theta)X_\theta^\perp(e^{-(D_\theta a)(z)}H_a I_{a,\theta}f)(s(ze^{-i\theta}),\theta))d\theta$$

gives an exact reconstruction formula for the density f *based on the data* $I_{a,\theta}f$ *of attenuated ray transforms of* f *over the integral curves of* X_θ

6. **Examples.** We present some worked examples of our method to familiar geometries. We stick to applications of Theorem 5.1 with similar formulae appearing for the attenuated data and applying Theorem 5.2.

Euclidean Lines. Earlier, we saw an inversion formula for the ray transform on Euclidean lines. We consider $f \in C_c^\infty(D^+)$ and parameterize the plane by $z(t,s) = t + is$. Then the complexified vector field is given by

$$X_\lambda = \lambda\frac{\partial}{\partial z} + \frac{1}{\lambda}\frac{\partial}{\partial \bar z}$$

$\xi(z,\lambda) = \lambda$ is holomorphic and has a zero of order 1 at $\lambda = 0$, $\rho(z,\lambda) = \frac{1}{\lambda}$ is meromorphic and has a simple pole at $\lambda = 0$ and the ratio $\frac{\xi}{\rho} = \lambda^2$ is analytic with a double root at the origin. Therefore X_λ is **type H**. Since $s(z,\lambda) = \frac{1}{2i}(\frac{z}{\lambda} - \bar z\lambda)$ is meromorphic in λ as are $\frac{\partial s(z,\lambda)}{\partial z} = \frac{1}{2i\lambda}$ and $\frac{\partial s(z,\lambda)}{\partial \bar z} = -\frac{\lambda}{2i}$ we see that $s(z,\lambda)$ is suitable. Thus, we may apply Theorem 5.1.

We recall remarks made at the beginning of section 3 regarding a sign convention on the transverse coordinate and that, as it now stands $X_\lambda^\perp s(z,\lambda) < 0$, therefore we must actually choose $X_\lambda^\perp = -\lambda_*\boldsymbol{\theta}^\perp \cdot \nabla$ instead of $\lambda_*\boldsymbol{\theta}\cdot\nabla$ in formula (29). Plugging this expression in then gives the following familiar reconstruction formula

$$f(z) = -\frac{1}{4\pi}\int_0^{2\pi} P(0,\theta)X_\theta^\perp H(If)(s(ze^{-i\theta}),\theta)d\theta$$

With $P(0,\theta) = 1$, the above reduces to formula (8) which we saw earlier.

Next, since $u(z,0) = 0$ we may apply the Corollary 1 to get the following inversion formula for the attenuated ray transform on straight lines in Euclidean space

$$f(z) = \frac{1}{4\pi}\int_0^{2\pi} \boldsymbol{\theta}^\perp \cdot \nabla\{e^{-(D_\theta a)(z)}H_a(I_{a,\theta}f)(s(ze^{-i\theta}),\theta)\}d\theta$$

which is in agreement with known results.

The Poincaré Disc. The unitized geodesics of the negatively curved hyperbolic disc are generated [6] by the following vector field

$$X|_z = (1-|z|^2)(\frac{1-z}{1-\bar z}\frac{\partial}{\partial z} + \frac{1-\bar z}{1-z}\frac{\partial}{\partial \bar z}), \qquad z \in D^+$$

This complexifies to the following

$$X_\lambda = (1-|z|^2)(\frac{\lambda - z}{1-\lambda\bar z}\frac{\partial}{\partial z} + \frac{1-\lambda\bar z}{\lambda - z}\frac{\partial}{\partial \bar z})$$

We see that $\xi(z,\lambda) = (1-|z|^2)\frac{\lambda - z}{1-\bar z\lambda}$ is analytic in λ and has a zero at $\lambda(z) = z$, $\rho(z,\lambda) = (1-|z|^2)\frac{1-\lambda\bar z}{\lambda - z}\frac{\partial}{\partial \bar z}$ is meromorphic and has no zeros in the disc, and the ratio $\frac{\xi}{\rho} = (\frac{\lambda - z}{1-\bar z\lambda})^2$ is complex-analytic in λ and has a double root at $\lambda = z$. Therefore X_λ is **type H**.

Moreover, the transverse coordinate s can be shown to be determined by $s(z,\lambda) = \frac{1}{2i}(\frac{1}{1-\lambda\bar{z}} - \frac{\lambda}{\lambda-z})$. Thus $s(z,\lambda)$, $\frac{\partial s(z,\lambda)}{\partial z} = -\frac{\lambda}{2i(\lambda-z)^2}$ and $\frac{\partial s(z,\lambda)}{\partial \bar{z}} = \frac{\lambda}{2i(1-\lambda\bar{z})^2}$ are each meromorphic in $\lambda \neq z$. Clearly $s(z,\lambda)$ is analytic in a neighborhood of the unit circle. Furthermore, since at $\lambda = z$, $s_{\bar{z}}(z,\lambda)$ picks up a $\delta(\lambda - z)$ term we have that $X_\lambda \mid_{\lambda=z} s(z,\lambda) = -\frac{\lambda\delta(\lambda-z)}{2i}$ and therefore $\frac{X_\lambda s(z,\lambda)}{(s(z,\lambda)-s(z_0,\lambda))^2} \sim -\frac{\lambda\delta(\lambda-z)}{2i(s(z,\lambda)-s(z_0,\lambda))^2} \sim \frac{\lambda\delta(\lambda-z)(\lambda-z)^2}{2ig_\lambda(z,z_0)} = 0$. An easy calculation shows that $\frac{s_z(z_0,\lambda)}{s(z,\lambda)-s(z_0,\lambda)}$ stays bounded for $z \neq z_0$ and $\lambda \in D^+$. Therefore, $s(z,\lambda)$ is suitable and one has, on applying Theorem 5.1 that

$$f(z) = \frac{1}{4\pi}\int_0^{2\pi} P(z,\theta)\{(1-|z|^2)(\frac{e^{i\theta}-z}{1-e^{i\theta}\bar{z}}\frac{\partial}{\partial z} + \frac{1-e^{i\theta}\bar{z}}{e^{i\theta}-z}\frac{\partial}{\partial \bar{z}})\}HI_\theta f(s(ze^{-i\theta}),\theta)d\theta$$

which agrees with what was obtained in [6] (see equation (80) of that article) using techniques of Riemann-Hilbert theory.

Furthermore, because $\lim_{\lambda\to z}|s(z,\lambda)| = \infty$, we notice that $u(z,\lambda(z)) = 0$ and we may apply Corollary 1 to get that

$$f(z) = \frac{1}{4\pi}\int_0^{2\pi} P(\lambda_i,\theta)X_\theta^\perp(e^{-(D_\theta a)(z)}H_a I_{a,\theta} f)(s(ze^{-i\theta}),\theta))d\theta \tag{33}$$

which agrees with formula (92) in [6] and gives a full reconstruction formula for the attenuated ray transform along the geodesics of the Poincaré disc.

The Spherical Cap. We consider an extension of the method so far presented to the case where the first condition of **type H** vector field is not satisfied. We consider the restriction of great circles, geodesics of S^2, onto the upper hemisphere $H^+ = \{\mathbf{x} \in \mathbb{R}^3; (x^1)^2 + (x^2)^2 + (x^3)^2 = 1, \quad x^3 > 0\}$. Consider the projection of these curves onto the unit disc which we may parameterize via the mapping $\boldsymbol{\gamma}(t,s) : \mathbb{R}^2 \to D^+$ defined by $\boldsymbol{\gamma}(t,s) = (x(t,s), y(t,s))$ where

$$x(t,s) = \frac{t}{\sqrt{1+t^2+s^2}}, \qquad y(t,s) = \frac{-s}{\sqrt{1+t^2+s^2}}$$

With $z = t + is$, a calculation reveals that $X_1 = \boldsymbol{\gamma}_*\frac{\partial}{\partial t} = \sqrt{1-|\mathbf{x}|^2}\{(1-x^2)\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}\}$. The complexification $X_\lambda = \lambda_* X_1$ of the vector field is therefore equal to the following

$$X_\lambda = \frac{\sqrt{1-|z|^2}}{2}\{\lambda(2 - \frac{z^2}{\lambda^2} - |z|^2)\frac{\partial}{\partial z} + \frac{1}{\lambda}(2 - \lambda^2\bar{z}^2 - |z|^2)\frac{\partial}{\partial \bar{z}}$$

We remark that the above obviously fails to be **type H** since $\xi(z,\lambda) = \frac{\sqrt{1-|z|^2}}{2}\lambda(2 - \frac{z^2}{\lambda^2} - |z|^2)$ is singular at $\lambda = 0$. A careful check of the arguments given in the last section reveals that the only place where holomorphy of ξ was really necessary in deriving Theorem 5.1 was in ensuring holomorphy of the term $\xi(z,\lambda)u_z(z,\lambda)$. Therefore, if we can establish this, we may proceed as before.

Since $\frac{\xi(z,\lambda)}{\rho(z,\lambda)} = \frac{\lambda^2 - \frac{z^2}{2-|z|^2}}{1-\lambda^2\frac{\bar{z}}{2-|z|^2}}$, we see that $\frac{\xi(z,\lambda)}{\rho(z,\lambda)}$ is complex-analytic in the disc $\lambda \in D^+$ for each z with zeros it shares with $\xi(z,\lambda)$ located at $\lambda_1(z) = \frac{z}{\sqrt{2-|z|^2}}$, $\lambda_2(z) = -\frac{z}{\sqrt{2-|z|^2}}$. Clearly $\rho(z,\lambda)$ is meromorphic. Therefore, aside from the fact that $|\xi(z,0)| = \infty$, one has that the vector field is of **type H**.

Moreover, clearly $s(z,\lambda) = \frac{\bar{z}\lambda - \frac{z}{\lambda}}{2i\sqrt{1-|z|^2}}$ is, for each $z \in D^+$, meromorphic in λ, with a Taylor expansion holding near $|\lambda| = 1$. A calculation shows that

$$(34) \qquad \frac{\partial s(z,\lambda)}{\partial z} = -\frac{2 - \bar{z}^2\lambda^2 - |z|^2}{4i\lambda(1-|z|^2)^{\frac{3}{2}}}, \qquad \frac{\partial s(z,\lambda)}{\partial \bar{z}} = \lambda\frac{2 - \frac{z^2}{\lambda^2} - |z|^2}{4i(1-|z|^2)^{\frac{3}{2}}}$$

for all $(z,\lambda) \in D^+ \times D^+$. This being the case, one clearly has that $X_\lambda s(z,\lambda) = 0$ holds everywhere and the first three conditions of suitability for s are established. We only need to check the boundedness of $\frac{s_z(z,\lambda)}{s(z,\lambda)-s(z_0,\lambda)}$ for $z \neq z_0$. For this we verify that

$$\frac{s_z(z,\lambda)}{s(z,\lambda) - s(z_0,\lambda)} = -\frac{2 - \bar{z}^2\lambda^2 - |z|^2}{2((\bar{z}\lambda^2 - z)(1-|z|^2) - (z_0\lambda^2 - z_0)\frac{(1-|z|^2)^{\frac{3}{2}}}{\sqrt{1-|z_0|^2}}}$$

bounded in $\lambda \in D^+ \setminus \{0\}$ and whereby moreover for $|\lambda| \to 0^+$ we have that $\frac{s_z(z,\lambda)}{s(z,\lambda)-s(z_0,\lambda)} \to \frac{2-|z|^2}{2(z(1-|z|^2) - \frac{z_0(1-|z|^2)^{\frac{3}{2}}}{\sqrt{1-|z_0|^2}})}$ which is integrable for $z \neq z_0$. Then, it follows that $s(z,\lambda)$ is suitable. More tellingly, we investigate $G_\lambda(z;z_0)$ and find that, on using formula (22) one has

$$(35) \qquad G_\lambda(z;z_0) = -\frac{1}{2\pi i(1-|z_0|^2)^2\{\frac{\bar{z}\lambda - \frac{z}{\lambda}}{\sqrt{1-|z|^2}} - \frac{\bar{z}_0\lambda - \frac{z_0}{\lambda}}{\sqrt{1-|z_0|^2}}\}}$$

Therefore $u(z,\lambda) \to 0$ as $|\lambda| \searrow 0$ and we verify that

$$\frac{\partial}{\partial z}G_\lambda(z,z_0) = \frac{2 - \bar{z}^2\lambda^2 - |z|^2}{8\lambda\pi(1-|z|^2)^{\frac{3}{2}}(1-|z_0|^2)^2\{\frac{\bar{z}\lambda - \frac{z}{\lambda}}{\sqrt{1-|z|^2}} - \frac{\bar{z}_0\lambda - \frac{z_0}{\lambda}}{\sqrt{1-|z_0|^2}}\}^2}$$

From which we conclude that $\xi(z,\lambda)\frac{\partial u(z,\lambda)}{\partial z}$ stays bounded as $\lambda \to 0$ and therefore is complex-analytic on all of D^+.

Since the behavior of $\xi(z,\lambda)u_z(z,\lambda)$ was our chief obstacle, we may then proceed as usual and apply our Theorem 5.1 to get

$$(36) \qquad f(z) = \frac{\sqrt{1-|z|^2}}{8\pi}\int_0^{2\pi} P(\frac{z}{\sqrt{2-|z|^2}},\theta)X_\theta^\perp HI_\theta f(s(ze^{-i\theta}),\theta)d\theta$$

where $X_\theta^\perp = i\{-e^{i\theta}(2 - e^{-2i\theta}z^2 - |z|^2)\frac{\partial}{\partial z} + e^{-i\theta}(2 - e^{2i\theta}\bar{z}^2 - |z|^2)\frac{\partial}{\partial \bar{z}}\}$. In the above expression, we have used the root $\lambda_1(z)$.

We remark that

$$\begin{aligned}
&\int_0^{2\pi} P(\lambda_2(z),\theta)X_\theta^\perp HI_\theta f(s(ze^{-i\theta}),\theta)d\theta \\
&\qquad = \int_0^{2\pi} P(-\lambda_1(z),\theta)X_\theta^\perp HI_\theta f(s(ze^{-i\theta}),\theta)d\theta \\
&\qquad = \int_\pi^{3\pi} P(\lambda_1(z),\omega)X_{\omega-\pi}^\perp HI_\theta f(-s(ze^{-i\omega}),\omega-\pi)d\omega \\
&\qquad = \int_0^{2\pi} P(\lambda_1(z),\omega)X_\omega^\perp HI_\theta f(s(ze^{-i\omega}),\omega)d\omega
\end{aligned}$$

where, in the above, we have used the symmetry $X_\theta^\perp I_\theta f(s(ze^{-i\theta}),\theta)$ equals $X_{\theta-\pi}^\perp I_\theta f(-s(ze^{-i\theta}),\theta-\pi)$. We therefore see a redundancy in the choice of $\lambda_i(z)$ appearing in expression (36).

We do not have a vanishing of $u(z,\lambda_i(z))$ and therefore, the inversion of the attenuated ray transform on these curves is determined modulo this value and is given by an application of Theorem 5.2 as

$$f(z) - a(z)\Re\{u(z,\frac{z}{\sqrt{2-|z|^2}})\} =$$

$$\frac{\sqrt{1-|z|^2}}{8\pi}\int_0^{2\pi} P(\frac{z}{\sqrt{2-|z|^2}},\theta)X_\theta^\perp e^{-(D_\theta a)(z)}H_a I_{a,\theta} f(s(ze^{-i\theta}),\theta)d\theta \tag{37}$$

with H_a defined in equation (31). To the authors' knowledge formulas (36) and (37) are new.

7. **Conclusions.**

We have illustrated that the method of complexification of vector fields presented in this article allows for a compact unification of the inversion formulae given for ray transforms on both Euclidean space [27] and the Poincaré hyperbolic disc [6]. Extending the class of vector fields amenable to the aforementioned scheme beyond those of **type H** remains an open problem. Since the analyticity properties of the coefficients of the vector fields, ensured by the **condition H** and suitability of the transverse coordinate, justify the holomorphy of the Green's function it is unclear how one could alter the method in the absence of such conditions, although the recent article [32] may yield some insight. There also remains the question of finding sufficient (or even necessary) conditions on the *initial* vector field being holomorphic after the complexification scheme used above. Real-analyticity is perhaps the simplest necessary condition, but presumably there are much more stringent ones.

8. **Acknowledgments.**

The authors wish to kindly thank the anonymous referee for catching several typos on a careful reading of the first draft and whose comments greatly improved the overall readability and presentation of this article.

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Received xxxx 20xx; revised xxxx 20xx.