1 Dot Product, Cauchy-Schwarz, Etc

1. Verify that $||\mathbf{v}|| \geq 0$ with $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.
2. Let $||\mathbf{v}|| = 2$ and $||\mathbf{w}|| = 4$ and $\mathbf{w} \cdot \mathbf{v} = 2$. Compute $||2\mathbf{v} + 17\mathbf{w}||$.
3. Let $||\mathbf{v}|| = 2$ and $||\mathbf{w}|| = 3$ and $\mathbf{v} \cdot \mathbf{w} = 5$. Compute $||12\mathbf{v} - 7\mathbf{w}||$.
4. In introducing the Cauchy-Schwarz inequality, it was alleged to generalize the dot product formula involving cosine which we found in $\mathbb{R}^2$. Explain how.
5. Explain why the triangle inequality has its name.
6. Prove that $d(c\mathbf{v}, \mathbf{w}) = |c| d(\mathbf{v}, \frac{1}{c}\mathbf{w})$ holds for all $0 \neq c \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$
7. Show that $\mathbf{v} \cdot \mathbf{w} = \frac{1}{4}(||\mathbf{v} + \mathbf{w}||^2 - ||\mathbf{v} - \mathbf{w}||^2)$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.
8. Show that $||\mathbf{v}||^2 + ||\mathbf{w}||^2 = \frac{1}{2}(||\mathbf{v} + \mathbf{w}||^2 + ||\mathbf{v} - \mathbf{w}||^2)$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.
9. Pythagorean theorem in $\mathbb{R}^n$. Namely, if $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ then $||\mathbf{v} + \mathbf{w}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2$ if and only if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.
10. True/False: $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are orthogonal if and only if $||\mathbf{v} - \mathbf{w}|| = ||\mathbf{v} + \mathbf{w}||$
11. True/False: $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w} \in \mathbb{R}^n$ are orthogonal if and only if $||\mathbf{v}|| = ||\mathbf{w}||$
12. (Requires linear independence). Suppose $S = \{\mathbf{v}_1, ..., \mathbf{v}_k\} \subset \mathbb{R}^n$ is a collection of nonzero mutually orthogonal vectors, i.e. $i \neq j \implies \mathbf{v}_i \cdot \mathbf{v}_j = 0$. Prove that the vectors in S are linearly independent.
13. True/False: All linearly independent sets of vectors in $\mathbb{R}^n$ are orthogonal.
14. Show that,[1] for all non-negative numbers $x, y \in \mathbb{R}$ we have $\sqrt{xy} \leq \frac{1}{2}(x + y)$.
15. If $a_1, ..., a_n$ are real numbers then
$$\frac{|a_1 + \cdots + a_n|}{\sqrt{n}} \leq \sqrt{a_1^2 + \cdots + a_n^2}$$
16. Let $a_1, ..., a_n$ be real and $b_1, ..., b_n$ be *positive*. Then prove that
$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + \cdots + a_n)^2}{b_1 + \cdots + b_n}$$
and that equality holds when and only when $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.
17. Let x, y, z be positive numbers. Prove that
$$\sqrt{x(3x + y)} + \sqrt{y(3y + z)} + \sqrt{z(3z + x)} \leq 2(x + y + z)$$
18. Find $\mathbf{proj}_{\begin{bmatrix} 1 & 2 \end{bmatrix}^{\mathsf{T}}} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$
19. Find $\mathbf{proj}_{\begin{bmatrix} 2 & 1 & 2 \end{bmatrix}^{\mathsf{T}}} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$
20. Find $\mathbf{proj}_{\begin{bmatrix} 3 & -1 & 2 \end{bmatrix}^{\mathsf{T}}} \begin{bmatrix} -3 \\ 4 \\ 7 \end{bmatrix}$

[1] The left hand side of the inequality is called the *geometric mean* of the numbers x, and y, whereas the right hand side is the *arithmetic mean* of the same numbers. This result shows that the geometric mean is bounded by the arithmetic mean. Sometimes this fact is called the "AM-GM inequality".

21. Let $\mathbf{v}, \in \mathbb{R}^n$ and $\neq \mathbf{0}$. Show that $\mathbf{proj_d}(\mathbf{proj_d v}) = \mathbf{proj_d v}$

22. Let $\mathbf{0} \neq \mathbf{d} \in \mathbb{R}^n$. Prove that $\mathbf{proj_d v} = \mathbf{0}$ if and only if $\mathbf{v}$ and $\mathbf{d}$ are orthogonal.

23. (Harder). Suppose $S = \{\mathbf{v}_1, ..., \mathbf{v}_k\} \subset \mathbb{R}^n$ is an orthogonal set. Let $\mathbf{v} \in \mathbb{R}^n$ be arbitrary and let c_i be the component of $\mathbf{v}_i$ along $\mathbf{v}$ i.e. $c_i = \frac{\mathbf{v}_i \cdot \mathbf{v}}{||\mathbf{v}_i||^2}$ for $i = 1, ..., k$. Show that

$$||\mathbf{v} - (c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k)|| \leq ||\mathbf{v} - (a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k)||$$

holds for all $a_1, ..., a_k \in \mathbb{R}$

24. (Hard). Suppose $S = \{\mathbf{v}_1, ..., \mathbf{v}_k\} \subset \mathbb{R}^n$ is an orthonormal set. Let $\mathbf{v} \in \mathbb{R}^n$ be arbitrary and let c_i be the component of $\mathbf{v}_i$ along $\mathbf{v}$ i.e. $c_i = \frac{\mathbf{v}_i \cdot \mathbf{v}}{||\mathbf{v}_i||^2}$ for $i = 1, ..., k$. Show that

$$c_1^2||\mathbf{v}_1||^2 + \cdots + c_k^2||\mathbf{v}_k||^2 \leq ||\mathbf{v}||^2$$