Homework Questions not in the Textbook

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1 General Knowledge/Sets

1. Describe the sets $\{x \in \mathbb{R} \mid \frac{x}{2} = k, \quad k \in \mathbb{N}\}$ and $\{x, y \in \mathbb{R} \mid \frac{x}{y} = 2, y \neq 0\}$
2. Describe the set $\{x \in \mathbb{N} \mid x \neq kl, k, l \in \mathbb{N}, k < x, l < x, k \neq 1, l \neq 1\}$
3. Write down a definition of $S_1 \cap S_2$ using set notation.
4. Show that $(S_1 \cap S_2)^c = S_1^c \cup S_2^c$ holds for all sets S_1 and S_2.
5. Show that $(S_1 \cup S_2)^c = S_1^c \cap S_2^c$ holds for all sets S_1 and S_2.
6. Prove that $(S_1 \cap S_2) \subset (S_1 \cup S_2)$ holds for all sets S_1 and S_2.
7. Are the notations $\{s\} \subset S$ and $s \in S$ interchangeable? Why or why not?
8. Prove that for any subset T of set S, we have $S = T \cup T^c$
9. Here's a claim: $S = T \iff \#S = \#T$. If true prove it. If false, give a counterexample.
10. Use induction to prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all positive integers n.
11. Use induction to prove that $2 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$ holds for all $n > 0$.
12. Use induction to prove that $7n < 8^{n+1}$ holds for $n \geq 0, n \in \mathbb{N}$
13. Use induction to prove the following claim: Every nonempty subset of $\mathbb{N}$ has a smallest element.
14. Consider a set S. We define a new set, $\mathcal{P}(S) = \{\text{all subsets of S}\}$.
    - Write down $\mathcal{P}(\{0, 1\})$.
    - Write down $\mathcal{P}(\{0\})$ and $\mathcal{P}(\mathcal{P}(\{0\}))$
    - Use induction to prove that $\#\mathcal{P}(S) = 2^{\#S}$ holds for all sets with $\#S < \infty$
15. (Harder). Consider the set $S = \{s \mid s \subseteq S, s \notin s\}$. Is this set well-defined? Why or why not?
16. (Harder). Use induction to prove that $8^n - 3^n$ is divisible by 5 for $n > 0$

17. (Harder). Consider a well-known *incorrect* use of induction used to “prove" the false claim that cars are all the same colour. The “proof" goes like this:

> The claim is that cars are all the same colour. This is equivalent to the claim that any set of cars must contain cars of the same colour. The claim is trivially true for the base case of $n = 1$ cars since, after all, a car has only one colour.[1] Next, we'll assume that the claim is true for all sets of n cars are we'll now consider a set of $n + 1$ cars, $\{\text{car } 1, \text{car } 2, \ldots, \text{car } n, \text{car } n+1\}$. We'll consider two subsets of this set $C_1 = \{\text{car } 1, \text{car } 2, \ldots, \text{car } n\}$ and $C_2 = \{\text{car } 2, \ldots, \text{car } n, \text{car } n+1\}$, each of which is a set of n cars. Therefore, by our induction hypothesis, C_1 and C_2 only contain cars of a single colour. But then again, since there's overlap in the entries of C_1 and C_2 the colour of each must be the same. Therefore $\{\text{car } 1, \text{car } 2, \ldots, \text{car } n, \text{car } n+1\}$ must have only cars of a single colour. Thus, cars are all of the same colour.

   What went wrong with the above “proof"?

18. (Harder). In a previous question you were asked to use induction to show that every nonempty subset of $\mathbb{N}$ has a smallest element. Now show the converse. Namely show that *if* the claim that every nonempty subset of $\mathbb{N}$ has a smallest element is a true fact about set the $\mathbb{N}$, *then* induction is a valid method of proof.

2 Systems of Equations

1. Give three examples of linear equations which have zero, one, and infinitely many solutions respectively.
2. Solve $x + 2y = 3$, leaving your answer in parameterized form.
3. Solve $x + 2y + 3z = 4$ leaving your answer in parameterized form.
4. Prove that the Elementary Manipulations of Systems result in equivalent systems.
5. Use Elementary Manipulations of Systems to solve the following, leaving your answer in parameterized form where appropriate.

   (a) $$\begin{aligned} 4x + y &= 0 \\ 16x + 4y &= 0 \end{aligned}$$

   (b) $$\begin{aligned} x + 2y &= 3 \\ 2x + 5y &= 6 \end{aligned}$$

   (c) $$\begin{aligned} 4x + y + 6z &= 0 \\ 5x + y &= 1 \end{aligned}$$

   (d) $$\begin{aligned} 4x + y + 6z &= 0 \\ 2x + 2y + 2z &= 9 \\ 5x + y &= 1 \end{aligned}$$

6. Suppose I have a system of three equations in three unknowns. Can such a system have exactly 3 distinct solutions? Can it have exactly 2 distinct solutions? Explain why or why not.
7. Show that the elementary row operations each have an *inverse* row operation of the same type. In other words, show that any of the elementary row operations can be undone by an elementary row operation of the same type.

[1] Though this isn't true for real cars, we'll actually pretend this is true for cars here. In other words this line *is not* where the mistake happens.

3 Dot Product, Cauchy-Schwarz, Etc

1. I claimed that $\mathbf{x}^T\mathbf{y} = \mathbf{x}\cdot\mathbf{y}$ cannot be *literally* true (though we shall often adopt the notation) on the grounds that $\mathbf{x}^T\mathbf{y}$ is a 1×1 matrix and $\mathbf{x} \cdot \mathbf{y}$ is a number. Is there actually a difference between 1×1 matrices and numbers? If so, what?

2. Verify that $||\mathbf{v}|| \geq 0$ with $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

3. Let $||\mathbf{v}|| = 2$ and $||\mathbf{w}|| = 4$ and $\mathbf{w} \cdot \mathbf{v} = 2$. Compute $||2\mathbf{v} + 17\mathbf{w}||$.

4. Let $||\mathbf{v}|| = 2$ and $||\mathbf{w}|| = 3$ and $\mathbf{v} \cdot \mathbf{w} = 5$. Compute $||12\mathbf{v} - 7\mathbf{w}||$.

5. In introducing the Cauchy-Schwarz inequality, it was alleged to generalize the dot product formula involving cosine which we found in $\mathbb{R}^2$. Explain how.

6. Explain why the triangle inequality (Theorem 2.2) has its name.

7. Prove that $d(c\mathbf{v}, \mathbf{w}) = |c|d(\mathbf{v}, \frac{1}{c}\mathbf{w})$ holds for all $0 \neq c \in \mathbb{R}$ and $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$

8. Show that $\mathbf{v} \cdot \mathbf{w} = \frac{1}{4}(||\mathbf{v} + \mathbf{w}||^2 - ||\mathbf{v} - \mathbf{w}||^2)$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

9. Show that $||\mathbf{v}||^2 + ||\mathbf{w}||^2 = \frac{1}{2}(||\mathbf{v} + \mathbf{w}||^2 + ||\mathbf{v} - \mathbf{w}||^2)$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

10. Pythagorean theorem in $\mathbb{R}^n$. Namely, if $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ then $||\mathbf{v} + \mathbf{w}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2$ if and only if $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

11. True/False: $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are orthogonal if and only if $||\mathbf{v} - \mathbf{w}|| = ||\mathbf{v} + \mathbf{w}||$

12. True/False: $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w} \in \mathbb{R}^n$ are orthogonal if and only if $||\mathbf{v}|| = ||\mathbf{w}||$

13. Suppose $S = \{\mathbf{v}_1, ..., \mathbf{v}_k\} \subset \mathbb{R}^n$ is a collection of nonzero mutually orthogonal vectors, i.e. $i \neq j \implies \mathbf{v}_i \cdot \mathbf{v}_j = 0$. Prove that the vectors in S are linearly independent.

14. True/False: All linearly independent sets of vectors in $\mathbb{R}^n$ are orthogonal.

15. Let A be an $m \times n$ matrix. Prove that $A^T A = I_n$ if and only if the columns of A are orthogonal.

16. Show that,[2] for all non-negative numbers $x, y \in \mathbb{R}$ we have $\sqrt{xy} \leq \frac{1}{2}(x + y)$.

17. If $a_1, ..., a_n$ are real numbers then

$$\frac{|a_1 + \cdots + a_n|}{\sqrt{n}} \leq \sqrt{a_1^2 + \cdots + a_n^2}$$

18. Let $a_1, ..., a_n$ be real and $b_1, ..., b_n$ be *positive*. Then prove that

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \cdots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + \cdots + a_n)^2}{b_1 + \cdots + b_n}$$

and that equality holds when and only when $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}$.

19. Let x, y, z be positive numbers. Prove that

$$\sqrt{x(3x + y)} + \sqrt{y(3y + z)} + \sqrt{z(3z + x)} \leq 2(x + y + z)$$

20. Find $\begin{bmatrix} 1 & 2 \end{bmatrix}^T \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

[2] The left hand side of the inequality is called the *geometric mean* of the numbers x, and y, whereas the right hand side is the *arithmetic mean* of the same numbers. This result shows that the geometric mean is bounded by the arithmetic mean. Sometimes this fact is called the "AM-GM inequality".

21. Find $\begin{bmatrix} 2 & 1 & 2 \end{bmatrix}^T \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$

22. Find $\begin{bmatrix} 3 & -1 & 2 \end{bmatrix}^T \begin{bmatrix} -3 \\ 4 \\ 7 \end{bmatrix}$

23. Let $\mathbf{v}, \in \mathbb{R}^n$ and $\neq \mathbf{0}$.

    (a) Show that $d(dv) = dv$

    (b) Use the above to find two examples of nonzero matrices, other than the identity, in $\mathbb{R}^3$ satisfying $A^2 = A$

24. Let $\mathbf{0} \neq \in \mathbb{R}^n$. Prove that $\mathbf{v} = \mathbf{0}$ if and only if $\mathbf{v}$ and are orthogonal.

25. (Harder). Suppose $S = \{\mathbf{v}_1, ..., \mathbf{v}_k\} \subset \mathbb{R}^n$ is an orthogonal set. Let $\mathbf{v} \in \mathbb{R}^n$ be arbitrary and let c_i be the component of $\mathbf{v}_i$ along $\mathbf{v}$ i.e. $c_i = \frac{\mathbf{v}_i \cdot \mathbf{v}}{\|\mathbf{v}_i\|^2}$ for $i = 1, ..., k$. Show that

$$\|\mathbf{v} - (c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k)\| \leq \|\mathbf{v} - (a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k)\|$$

    holds for all $a_1, ..., a_k \in \mathbb{R}$

26. (Harder). Suppose $S = \{\mathbf{v}_1, ..., \mathbf{v}_k\} \subset \mathbb{R}^n$ is an orthonormal set. Let $\mathbf{v} \in \mathbb{R}^n$ be arbitrary and let c_i be the component of $\mathbf{v}_i$ along $\mathbf{v}$ i.e. $c_i = \frac{\mathbf{v}_i \cdot \mathbf{v}}{\|\mathbf{v}_i\|^2}$ for $i = 1, ..., k$. Show that

$$c_1^2 + \cdots + c_k^2 \leq \|\mathbf{v}\|^2$$

4 Rank

1. Prove that spans of vectors are always subspaces.

2. True/False: If S_1 and S_2 are subspaces of $\mathbb{R}^n$ then so is $S_1 \cup S_2$. If true, prove it, otherwise give a counterexample.

3. True/False: If S_1 and S_2 are subspaces of $\mathbb{R}^n$ then so is $S_1 \cap S_2$. If true, prove it, otherwise give a counterexample.

4. Find $E_3(\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix})$ and $E_{-1}(\begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix})$.

5. Find $E_\lambda(I_n)$ for all values of $\lambda \in \mathbb{R}$.

6. Prove that invertible linear transformations map bases of $\mathbb{R}^n$ onto bases of $\mathbb{R}^n$. In other words if A is an invertible $n \times n$ matrix, prove that $\{A\mathbf{v}_j\}_{j=1}^n$ is a basis whenever $\{\mathbf{v}\}_{j=1}^n$ is.

7. Prove that if $A \sim B$ then $row(A) = row(B)$.

8. Prove that the reduction algorithm applied to $A\mathbf{x} = \mathbf{0}$ will produce a basis for $null(A)$ whenever $null(A) \neq \{\mathbf{0}\}$. *Hint: think about reduced row-echelon form.*

9. In the proof of Theorem 1.2, how do I know it's possible to choose a largest set of linearly independent vectors $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$?

10. It was claimed that the zero subspace admits no basis. Explain why.

11. Suppose that AB is invertible. Is A? If so prove it. If not, give a counterexample.

12. Suppose A, B are $n \times n$ matrices. Use rank inequalities to show that if A is not invertible then AB must be non-invertible as well.

13. Prove that if U, V are subspaces of $\mathbb{R}^n$ then $U \subseteq V$ implies $\dim U \leq \dim V$.

14. Use a theorem on maximal rank to prove that the polynomial $p(x) = 1 + x^2$ has no real roots.

15. Let $A = \begin{bmatrix} 2 & 2 & 3 \end{bmatrix}$. Find a basis for $\ker(A)$ and a basis for $\ker(A^T)$.

16. Suppose that A, B are square matrices of the same size and that $col(B) \subseteq \ker A$. What can I say about AB?

17. Let $A = [a_{ij}]$ where $a_{ij} = i + j$ for $1 \leq i, j \leq n$. Determine $rank(A)$.

18. Suppose that A is $m \times n$. Show that $\ker A = \ker(UA)$ for all invertible $m \times m$ matrices U. Then also show that $\dim(\ker(A)) = \dim(\ker(AB))$ for all invertible $n \times n$ matrices B.

19. Let
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$
    (a) Find a matrix B such that $AB = I_2$ or show that such a B doesn't exist.
    (b) Find a matrix C such that $CA = I_3$ or show that such a C doesn't exist.

20. Let
$$A = \begin{bmatrix} a & 1 & a & 0 & 0 & 0 \\ 0 & b & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & c & 0 \\ 0 & 0 & 0 & d & 1 & d \end{bmatrix}$$
    where a, b, c, d are unspecified real numbers.
    (a) Prove that $\text{rank}(A) > 2$.
    (b) Prove that if $a = d = 0$ and $bc = 1$ then $\text{rank}(A) = 3$.

21. Let A be an $n \times n$ matrix.
    (a) Show that $A^2 = 0$ if and only if $col(A) \subseteq \ker(A)$.
    (b) Show that $A^2 = 0$ implies $rank(A) \leq \frac{n}{2}$
    (c) Find a matrix A such that $col(A) = null(A)$

22. Let $\mathbf{c} \neq \mathbf{0}$ be in $\mathbb{R}^m$ and $\mathbf{r} \in \mathbb{R}^n$ and define $A = \mathbf{c}\mathbf{r}^T$
    (a) Show that $col(A) = \text{span}\{\mathbf{c}\}$ and $row(A) = \text{span}\{\mathbf{r}\}$.
    (b) Find $\dim(\ker(A))$
    (c) Show that $\ker A = null(\mathbf{r})$

23. Prove that if $\dim S = m$ then any set of m linearly independent vectors in S must be a basis.

24. Prove that if $\dim S = m$ then any set of m spanning vectors in S must be a basis.

25. Prove that the only *proper* subspaces of $\mathbb{R}^3$ are lines through the origin and planes through the origin.

26. Prove the remaining equivalent statements in the theorems on full rank (Theorems 9 & 10).

27. Let B be $m \times n$ and AB be $k \times n$. Suppose that $rank(AB) = rank(B)$. Show that $\ker B = \ker(AB)$.

28. Show that, for every $n \times m$ matrix A, the matrix $I_m + A^T A$ is invertible.

29. Suppose that $S \subset \mathbb{R}^8$ where $\dim S = 5$. Is there a subspace $V \subseteq \mathbb{R}^8$ such that $\dim V = 2$ and $V \cap S = \{\mathbf{0}\}$. Justify your answer.

5 Fundamental Theorem of Linear Algebra

1. Calculate the orthogonal complements of the improper subspaces of $\mathbb{R}^n$.
2. Let $\mathbf{v} \in \mathbb{R}^3$ be a nonzero vector. Describe $(\text{span}\{\mathbf{v}\})^\perp$.
3. Prove that $S^\perp$ is a subspace of $\mathbb{R}^n$ whenever S is.
4. Prove that for $S \subseteq \mathbb{R}^n$ a subspace we have $\dim S + \dim S^\perp = n$.
5. If S, W are subspaces of $\mathbb{R}^n$ show that $S \subseteq W \implies W^\perp \subseteq S^\perp$
6. Let $S = \{\mathbf{x} \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}$. Prove that S is a subspace and find a basis for $S^\perp$.
7. Prove that every $m \times n$ matrix A defines a linear transformation from $row(A)$ **onto** $col(A)$, i.e.

$$T_A : row(A) \to col(A)$$

8. Consider A, an $m \times n$ matrix and $\mathbf{b} \in \mathbb{R}^m$. Consider the following claim[3]: One and only one of the following systems is solvable

   (a) $A\mathbf{x} = \mathbf{b}$

   (b) $A^T\mathbf{y} = \mathbf{0}$ with $\mathbf{y}^T\mathbf{b} \neq \mathbf{0}$

   Prove that the above options *cannot BOTH hold*. In other words, if one of the above holds the other one mustn't.

[3] A variation of this strange-sounding claim is very important in numerous applications including, very importantly, in differential equations.