Extra Exercises

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These should challenge you a bit beyond what you've done in textbook exercises. A few of them, which are marked, are quite challenging.

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1 True/False

1. Let A be a matrix. The pivot columns of the reduced row echelon form of A are a basis for the row space of the transpose of A.
2. If A is invertible then $rank(A) = rank(A^{-1})$.
3. The rank of a matrix must be at least as big as the dimension of its nullspace.
4. A nonzero element of $Nul(A)$, for a square matrix A, must be an eigenvector of A.
5. If every entry of a square matrix A is positive then $\det(A) \neq 0$.
6. Let A be $m \times k$ and B be $k \times n$ matrices which have reduced row echelon matrices given by matrices M and N respectively. Then the reduced row echelon matrix of AB is given by the matrix MN.
7. Let A be invertible. Then, if λ is an eigenvalue of A, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.
8. If A and B are 2×2 then $\det(2A + 3B) = 4\det(A) + 9\det(B)$.
9. If A is $n \times n$ and has eigenvalues $\lambda_1, ..., \lambda_k$ counted with multiplicities $m_1, ..., m_k$, then $\det(A) = \lambda_1^{m_1} \cdots \lambda_k^{m_k}$.
10. If A is square and $A^T = A^{-1}$ then $\det(A) = 1$.

11. Eigenvectors are linearly independent.
12. If $A\mathbf{x} = \mathbf{0}$ has only the trivial solution then A is invertible.
13. If A is invertible, then so must be its reduced row echelon form.
14. The circle $x^2 + y^2 = 1$ is the null space of some linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$.
15. The line $y = 6x$ is the null space of some linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$.
16. The line $y = -6x$ is the range of some linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$.
17. If A is 2×2 then $rank(A) = rank(A^2)$.
18. If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly independent vectors in subspace S then $\dim(S) > 3$.
19. Let A and B be $n \times n$. Then A and B are invertible if and only if AB is invertible.
20. If the characteristic polynomial of A is $c_A(\lambda) = (\lambda - 4)^3(\lambda + 5)$ then A is 4×4.
21. If the characteristic polynomial of A is $c_A(\lambda) = \lambda(\lambda - 1)^2(\lambda + 2)$ then there are no solutions to $A\mathbf{x} = 3\mathbf{x}$.
22. If 0 is an eigenvalue of A then A is not diagonalizable.
23. A is invertible if and only if $row(A) = col(A)$.
24. If A and B are the same size and each has rank 1 then $rank(A + B) \geq 1$.
25. If A and B are the same size and each have rank n then $rank(A + B) = n$.

2 Long Answer

1. (Some hard parts). This problem concerns planes and should challenge your intuitions.
   - A plane in $\mathbb{R}^3$ is the solution set to an equation of the form $ax+by+cz = d$. Prove that two planes in $\mathbb{R}^3$ have either zero or infinitely many intersection points.
   - A plane Π in $\mathbb{R}^4$ is a set of the form $\Pi = \{\mathbf{p} + s\mathbf{v}_1 + t\mathbf{v}_2 \mid s, t \in \mathbb{R}\}$, where $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent. Prove that two planes in $\mathbb{R}^4$ can meet at a single point.
   - Suppose two planes Π_1, Π_2 in $\mathbb{R}^4$ meet in a line L. Prove that there exists a translation[1] of one of the planes causing Π_1 and Π_2 to separate.
2. Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

[1]A *translation* of a plane $\Pi = \{\mathbf{p} + s\mathbf{v}_1 + t\mathbf{v}_2 \mid s, t \in \mathbb{R}\}$ by the vector $\mathbf{u}$ is the set $\{\mathbf{u} + \mathbf{p} + s\mathbf{v}_1 + t\mathbf{v}_2 \mid s, t \in \mathbb{R}\}$, often simply denoted $\mathbf{u} + \Pi$. It is a shift of all points in the plane by the vector $\mathbf{u}$.

(a) Find a matrix B such that $AB = I_2$ or show that such a B doesn't exist.

(b) Find a matrix C such that $CA = I_3$ or show that such a C doesn't exist.

3. Let

$$A = \begin{pmatrix} a & 1 & a & 0 & 0 & 0 \\ 0 & b & 1 & b & 0 & 0 \\ 0 & 0 & c & 1 & c & 0 \\ 0 & 0 & 0 & d & 1 & d \end{pmatrix}$$

where a, b, c, d are unspecified real numbers.

(a) Prove that $\text{rank}(A) > 2$.

(b) Prove that if $a = d = 0$ and $bc = 1$ then $\text{rank}(A) = 3$.

4. Let V be a subspace of dimension n and let $S = \{v_1, \ldots, v_k\}$ be a subset of V. Answer True/False to the following:

(a) If S is a basis for V then $k = n$.

(b) If S spans V then $k \leq n$.

(c) If S is linearly independent then $k \leq n$.

(d) If S is linearly independent and $k = n$ then S spans V.

(e) If S spans V and $k = n$ then S is a basis for V.

5. Find

$$\det\left(\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 & 3 \\ 4 & 3 & 2 & 1 & 2 \\ 5 & 4 & 3 & 2 & 1 \end{bmatrix}\right)$$

6. For this problem, we'll be considering a transformation. Let $\mathbf{d}$ be a fixed, nonzero vector in $\mathbb{R}^n$. Define the transformation $\mathbf{p_d} : \mathbb{R}^n \to \mathbb{R}^n$ by $\mathbf{p_d}(\mathbf{x}) = \frac{\mathbf{x} \cdot \mathbf{d}}{||\mathbf{d}||^2}\mathbf{d}$

(a) Prove that $\mathbf{p_d}$ is linear and find its standard matrix.

(b) Prove that $\mathbf{p_d} \circ \mathbf{p_d} = \mathbf{p_d}$

(c) Prove that $\mathbf{x} - \mathbf{p_d}(\mathbf{x})$ is orthogonal to $\mathbf{d}$ for all $\mathbf{x} \in \mathbb{R}^n$.

(d) Use part (b) to find an example of a nonzero 4×4 matrix, other than the identity, satisfying $A^2 = A$.

7. Are all matrices LU-factorizable? If not, provide an example of a matrix which is not factorable in this way.

8. A square matrix A is said to be *anti-symmetric* if $A^T = -A$.

(a) Give two examples of 2×2 anti-symmetric matrices.

(b) Prove that every square matrix B can be written as a sum of a symmetric and an anti-symmetric matrix. Namely, $B = S + A$ where $S^T = S$, $A^T = -A$.

9. The *adjugate* $adj(A)$ of a square matrix A is (uniquely) defined as the square matrix which satisfies $adj(A)A = Aadj(A) = \det(A)I$. Prove that $adj(A)^{-1} = adj(A^{-1})$ whenever A is invertible.

10. Let $\mathbf{c} \neq \mathbf{0}$ be in $\mathbb{R}^m$ and $\mathbf{0} \neq \mathbf{r} \in \mathbb{R}^n$ and define $A = \mathbf{c}\mathbf{r}^T$

    (a) Show that $col(A) = \text{span}\{\mathbf{c}\}$ and $row(A) = \text{span}\{\mathbf{r}\}$.

    (b) Find $\dim(Nul(A))$

    (c) Show that $Nul(A) = Nul(\mathbf{r}^T)$

11. Let A be an $n \times n$ matrix.

    (a) Show that $A^2 = 0$ if and only if $col(A) \subseteq \ker(A)$.

    (b) Show that $A^2 = 0$ implies $rank(A) \leq \frac{n}{2}$

    (c) Find a matrix A such that $col(A) = null(A)$

12. If the rows of a square matrix A all sum to the same number c, show that c is an eigenvalue of A.

13. If the columns of a square matrix A all sum to the same number c, show that c is an eigenvalue of A.

14. (Some hard parts). A square matrix is called *nilpotent* if there's an $m \geq 1$ such that $A^m = 0$.

    (a) Find some examples of nonzero nilpotent matrices.

    (b) Prove that nilpotent matrices have only 0 as an eigenvalue.

    (c) Prove that $c_A(x) = (-1)^n x^n$ for nilpotent matrices A.

    (d) If A, B are both nilpotent, of the same size, and $AB = BA$ holds, then prove AB and $A + B$ are both nilpotent as well.

    (e) Show that if A is nilpotent with $A^{r+1} = 0$, show that $I - A$ is invertible and that its inverse is $I + A + A^2 + \cdots + A^r$

15. Are all 1×1 matrices diagonalizable?

16. Diagonalize $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

17. Suppose A is diagonalizable.

(a) Is A^n diagonalizable for $n \geq 1$? If yes prove it, if no give a counterexample.

(b) Is cA diagonalizable for all $c \in \mathbb{R}$? If yes prove it, if no give a counterexample.

(c) Is MAM^{-1} diagonalizable for any invertible M with same size as A? If yes prove it, if no give a counterexample.

(d) Is $cI + A$ diagonalizable for all $c \in \mathbb{R}$? If yes prove it, if no give a counterexample.

18. Are all elementary matrices diagonalizable? Why or why not.

19. Show that, for any $n \times m$ matrix A, the matrix $I_m + A^T A$ is invertible. Give two separate proofs.

20. True/False: Every square matrix can be written as the sum of two invertible matrices. If false give a counterexample. If true, prove it.

21. Let B be $m \times n$ and AB be $k \times n$. Suppose that $rank(AB) = rank(B)$. Show that $\ker B = \ker(AB)$.

22. Let $c_A(x)$ be the characteristic polynomial for matrix A.

    (a) Show $c_{rA}(x) = r^n c_A(\frac{x}{r})$ for all $r \neq 0$

    (b) Show that $c_{A^2}(x^2) = c_A(x) c_A(-x)$

    (c) (*This may be quite tricky, but there may be an easy proof*) If $A^m = 0$ for some $m \geq 1$ show that $c_A(x) = x^n$

23. Let $A = [a_{ij}]$ be a square matrix with $a_{ij} = 1$ for $1 \leq i \leq n$ and $1 \leq j \leq n$. Is A diagonalizable? If yes, diagonalize it. If not, explain why not. *Hint: this can be done in a fast and easy way or a less fast and way less easy way. We suggest you choose the easy way.*

24. (Hard). Suppose that A is an $n \times n$ matrix which satisfies $A^2 - 3A + 2I = 0$.

    (a) Show that the only possible eigenvalues of A are $\lambda = 1, 2$.

    (b) Show that A must be diagonalizable.

    (c) Find all possible matrices satisfying $A^2 - 3A + 2I = 0$.

25. (Very hard). Suppose that A is an $m \times n$ matrix with n pivots. Show that there is an $n \times n$ matrix Q, with real entries, satisfying

$$Q^2 = A^T A$$

    Such a matrix Q would be called the *square root* of $A^T A$, namely $Q = \sqrt{A^T A}$.

26. (Some hard parts). Suppose that A is an $m \times n$ matrix with n pivots.

(a) Prove that there's a positive constant C such that $||A\mathbf{x}|| \leq C||\mathbf{x}||$ holds, for all $\mathbf{x} \in \mathbb{R}^n$.

(b) Verify that $\mathbf{x}^T A^T A\mathbf{x} = ||A\mathbf{x}||^2$

(c) Show that $A^T A$ is diagonalizable with *strictly positive* eigenvalues. Thus, $A^T A = PDP^{-1}$

(d) Prove there's a positive constant c such that $||P\mathbf{y}|| > c||\mathbf{y}||$ holds for all $\mathbf{y} \in \mathbb{R}^n$

(e) Show that there's a positive constant d such that $||D\mathbf{z}|| > d||\mathbf{z}||$ holds for all $\mathbf{z} \in \mathbb{R}^n$

(f) Conclude that there's a positive constant $\ell > 0$ such that $||A\mathbf{x}|| > \ell||\mathbf{x}||$ holds for all $\mathbf{x} \in \mathbb{R}^n$