MAT223 - Review Sheet

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Beginning on the next page,you'll find a handy list of theorems, definitions, and concepts we've used throughout the term which may aid your course review. The list was write by a former MAT223 prof K. Leung and given out during December 2016 in preparation for the final exam. Since this semester we covered a few different things from that term, a few important things are missing from this sheet, namely **pivots**, **LU Factorization**, **orthogonal complements** and **Fundamental Theorem of Linear Algebra**.[1] Please keep in mind that the material is somewhat incomplete when you read this.

[1]Plenty of material from this list was *removed* as well since there were things this term which we did not cover that semester. In fact, *more* was removed from the sheet than needs to be added in case you're curious.

linearly independent	A collection of k vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent if $$a_1\mathbf{v}_1 + \ldots + a_k\mathbf{v}_k = \mathbf{0} \text{ implies } a_1 = \ldots = a_k = 0.$$
linearly dependent	A collection of k vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly dependent if there is $a_1, \ldots, a_k \in \mathbb{R}$, not all zero, such that $$a_1\mathbf{v}_1 + \ldots + a_k\mathbf{v}_k = \mathbf{0}$$
span (noun)	The span of $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. $$\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \{a_1\mathbf{v}_1 + \ldots + a_k\mathbf{v}_k \mid a_1, \ldots a_k \in \mathbb{R}\}$$
span (verb)	$\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ spans a subspace U if every vector in U is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ and $\mathbf{v}_i \in U$ for all i, i.e. $U = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$
Basis	A collection of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a basis for a subspace U if $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent and it spans U.
Dimension	Dimension of a subspace U is the number of vectors in a basis for U.
Orthogonality	Two vectors $\mathbf{x}$ and $\mathbf{y}$ are orthogonal if $\mathbf{x} \cdot \mathbf{y} = 0$.
Orthogonal Sets	A collection of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthogonal set if $\mathbf{v}_i$ is orthogonal to $\mathbf{v}_j$ if $i \neq j$; and all vectors are nonzero.
null space	The null space of a matrix A is the set of vectors $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$
image space	The image space of a matrix A is all the vectors $\mathbf{v}$ such that $\mathbf{v}$ can be written as $A\mathbf{x}$ for some $\mathbf{x}$. It is exactly the same as the column space of A.
column space	The column space of a matrix A is the span of the column vectors of A. (It is exactly the same as the image space of A.
row space	The row space of a matrix A is the span of the row vectors of A.
rank	The rank of a matrix A is the number of leading 1's in its row-echelon form. It is also the dimension of the row space, column space and image space of A.
Cofactor	The (i, j)-th cofactor of a square matrix A is denoted by C_{ij} such that $C_{ij} = (-1)^{i+j}\det(A_{ij})$, where A_{ij} is the matrix by deleting the i-th row and the j-th column of A.
Eigenvalue	λ is an eigenvalue of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some **nonzero** $\mathbf{v}$.
Eigenvector	$\mathbf{v}$ is an eigenvector of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$ and $\mathbf{v} \neq \mathbf{0}$
Eigenspace	The eigenspace E_λ of A corresponding to an eigenvalue λ is the collection of all eigenvectors with eigenvalue λ, along with the zero vector. Thus $E_\lambda = \text{null}(\lambda I - A)$.
Characteristic polynomial	The characteristic polynomial $c_A(\lambda)$ of a square matrix A is defined as $c_A(\lambda) = \det(\lambda I - A)$.
Algebraic Multiplicity	The multiplicity m of an eigenvalue λ_0 is the number such that $c_A(\lambda) = (\lambda - \lambda_0)^m p(\lambda)$ for some polynomial $p(\lambda)$ such that $p(\lambda_0) \neq 0$.
Geometric Multiplicity	The geometric multiplicity of an eigenvalue λ is $\dim \text{null}(\lambda I - A)$

Theorems, propositions, lemmas, etc.

- Let U be a subspace. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ spans U and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is a linearly independent set in U, then $m \leq k$.

  In other words, the number of vectors in a linear independent set cannot be more that the number of vectors in a spanning set.

- If both $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ are bases for a subspace U, then $k = m$.

  In other words, for a fixed subspace, every basis contains the same number of vectors.

- Any independent set in U can be enlarged to a basis for U and any spanning set of U can be cut down to a basis for U.

- For any nonzero subspace U, U has a basis.

- Let U be a subspace such that $\dim U = m$. Then a set of m vectors in U is linearly independent if and only if it spans U.

- Let U and V be subspaces such that $U \subseteq V$. Then $\dim U \leq \dim V$.

- If U and V are subspaces such hat $U \subseteq V$ and $\dim U = \dim V$. Then $U = V$.

- Cauchy inequality: Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then $|\mathbf{x} \cdot \mathbf{y}| \leq ||\mathbf{x}||||\mathbf{y}||$, with equality holds when $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

- Triangle inequality: Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$.

- Every orthogonal set is linearly independent.

- If $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ is an orthogonal set, then $||\mathbf{x}_1 + \ldots + \mathbf{x}_k||^2 = ||\mathbf{x}_1||^2 + \ldots + ||\mathbf{x}_k||^2$.

- Row operations won't change the row space and column operations won't change the column space.

- $\text{rank}(A) = \text{rank}(A^T)$.

- $\text{col}(AB) \subseteq \text{col}(A)$ and $\text{row}(AB) \subseteq \text{row}(B)$.

- $\text{rank}(AB) \leq \text{rank}(A)$ and $\text{rank}(AB) \leq \text{rank}(B)$.

- Let A be an $m \times n$ matrix. Then $\text{rank}(A) + \dim \text{null}(A) = n$.

- The determinant of an $n \times n$ matrix A is given by

  $$\det(A) = a_{k1}C_{k1} + \ldots + a_{kn}C_{kn} = a_{1j}C_{1j} + \ldots + a_{nj}C_{nj},$$

  where j, k are any integer between 1 to n. This is the cofactor expansion of A.

- Properties of determinants

  - If A contains a zero row or column, then $\det A = 0$.

  - If two rows or columns are interchanged, the determinant of the resulting matrix is $-\det A$.
  - If a row or column of A is multiplied by a constant k, the deerminant of the resulting matrix is $k \det A$.
  - If two distinct rows or columns of A are identical, then $\det A = 0$.
  - If a multiple of a row (or column) of A is added to a different row (or column), then the determinant stays the same.
  - $\det A = \det A^T$
  - $\det[\mathbf{v}_1 \ \ldots \ \mathbf{b} + \mathbf{c} \ \ldots \ \mathbf{v}_n] = \det[\mathbf{v}_1 \ \ldots \ \mathbf{b} \ \ldots \ \mathbf{v}_n] + \det[\mathbf{v}_1 \ \ldots \ \mathbf{c} \ \ldots \ \mathbf{v}_n]$
  - $\det(AB) = \det(A)\det(B)$ (both are square matrices)

- A is invertible if and only if $\det A \neq 0$.

- Let λ be an eigenvalue of A with algebraic multiplicity m. Then we have $1 \leq \dim E_\lambda \leq m$

- Let A be an $n \times n$ matrix. Then A is diagonalizable if (a) the sum of the geometric multiplicities of all of its eigenvalues is n; and (b) for each eigenvalue λ, the algebraic multiplicity m equals $\dim E_\lambda$.

- If A has n distinct eigenvalues, then A is diagonalizable.

Algorithms

Prove that a collection of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly indepenedent.	• Write "Let $a_1\mathbf{v}_1 + \ldots + a_k\mathbf{v}_k = \mathbf{0}$". • Then manipulate the above to try to get $a_1 = \ldots = a_k = 0$.
Prove that a collection of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ spans $\mathbb{R}^n$	• Consider the matrix $A = [\mathbf{v}_1 \ \ldots \ \mathbf{v}_k]$. • Row reduce A to an echelon form R. • If R does not contain a zero row, then the collection of vectors spans $\mathbb{R}^n$. Otherwise the collection does not span $\mathbb{R}^n$.
Find a basis for the null space of A.	• Row reduce A to an echelon form R. • Solve the system $A\mathbf{x} = \mathbf{0}$. For any parameter column, write something like "Let $x_5 = s$, $x_4 = t$",etc. • Write down the basic solutions to the homoegeneous system $A\mathbf{x} = \mathbf{0}$ • The basic solutions is a basis for the null space of A.
Find a basis for the column space (or image space) of A.	• Row reduce A to an echelon form R. • Find the columns in R such that it contains leading 1's. • The *corresponding* columns of the original matrix A (not R) are a basis for the column space of A.
Find a basis for the row space of A.	• Row reduce A to an echelon form R. • The nonzero rows of R (not A) are a basis for the row space of A.
If $U = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, find a basis for U.	Method 1: • Let $A = [\mathbf{v}_1 \ \ldots \ \mathbf{v}_k]$. • Then U is the column space of A. Use the (above) method to find a basis for $\text{col}(A) = U$. Method 2: • Let $A = [\mathbf{v}_1 \ \ldots \ \mathbf{v}_k]^T$. • Then U is the row space of A. Use the (above) method to find a basis for $\text{row}(A) = U$.

Given an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ for U, express a vector $\mathbf{w} \in U$ in terms of a linear combination of the basis vectors	• Write $\mathbf{w} = a_1\mathbf{v}_1 + \ldots + a_k\mathbf{v}_k$. • Use the expansion formula (not covered this term) to get $a_i = \frac{\mathbf{w}\cdot\mathbf{v}_i}{\mathbf{v}_i\cdot\mathbf{v}_i}$.
Diagonalize A ($n \times n$)	• Solve the characteristic equation $\det(A - \lambda I) = 0$ for λ. • For each root λ, find a set of basic solutions the homogeneous system $(A - \lambda I)\mathbf{v} = \mathbf{0}$. • If the total number of basic solution vectors are n, then A is diagonalizable. • Put the basic solution vectors as column vectors to get P. Put their corresponding eigenvalues as entries in D in corresponding column. • Then $A = PDP^{-1}$.

Keywords, Definitions, Theorems, etc

- System of linear equations, augmented matrix, consistent, elementary row operations, reduction algorithm, row-echelon form, reduced row-echelon form
- Homogeneous system, basic solution, trivial solution
- matrix addition, multiplication, scalar multiplication, vectors, dot product, transpose, symmetric matrices
- $(AB)^T = B^T A^T$.
- A is invertible if there is B such that $AB = BA = I$.
- If A and B are invertible, then AB is also invertible, where $(AB)^{-1} = B^{-1}A^{-1}$.
- Elementary matrix: Performing elementary row operations is the same as multiplying by a corresponding elementary matrix on the left.
- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$; and $T(c\mathbf{x}) = cT(\mathbf{x})$ for all $c \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.
- Rotation transformation, reflection transformation.
- Vectors, tip-to-tail method, parallelogram law, parallel, orthogonal vectors
- Vector equation of a line. $\mathbf{x} = \mathbf{p} + t\mathbf{v}$, $t \in \mathbb{R}$. Here $\mathbf{p}$ is a point (as the position vector from the origin) and $\mathbf{v}$ is the direction vector.
- U is a subspace of $\mathbb{R}^n$ if (0) $\mathbf{0} \in U$; (1) If $\mathbf{x}, \mathbf{y} \in U$, then $\mathbf{x} + \mathbf{y}$ is also in U; and (2) If $\mathbf{x} \in U$ and $c \in \mathbb{R}$, then $c\mathbf{x}$ is also in U.

Algorithms

Solving a system $A\mathbf{x} = \mathbf{b}$.	• Method 1: Row reduce the augmented Matrix $[A\|\mathbf{b}]$ • Method 2: If A is an invertible square matrix, find A^{-1}. Then $\mathbf{x} = A^{-1}\mathbf{b}$.
Find A^{-1}.	• Method: Row reduce $[A\|I]$ to $[I\|A^{-1}]$.
If A is reduced to B by row operations, find U such that $B = UA$.	• Row reduce $[A\|I]$ to $[B\|U]$.
For a linear map T, find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.	• Use $A = [T(\mathbf{e}_1) \ \dots \ T(\mathbf{e}_k)]$

The Big Equivalence table

Let A be an $m \times n$ matrix with column vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. The followings are equivalent.

- The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- The homogeneous system $A\mathbf{x} = \mathbf{0}$ has a unique solution.
- Every column of A contains a pivot position.
- Every column of the echelon form of A contains a leading 1.
- The vector equation $x_1\mathbf{v}_1 + \ldots x_n\mathbf{v}_n = \mathbf{0}$ has only trivial solutions $x_1 = x_2 = \ldots x_n = 0$.
- The column vectors of A, namely $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent.
- null $A = \{\mathbf{0}\}$.
- $\text{rank}(A) = n$
- The rows of A span $\mathbb{R}^n$
- The $n \times n$ matrix A^TA is invertible
- $BA = I$ for some $n \times m$ matrix B.

$$\overset{rank n}{\Longleftarrow} \neq \overset{rank m}{\Longrightarrow}$$

- For every $\mathbf{b} \in \mathbb{R}^m$, the homogeneous system $A\mathbf{x} = \mathbf{b}$ is consistent.
- For every $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$.
- Every row of A contains a pivot position.
- There is no zero row in the echelon form of A.
- For every $\mathbf{b} \in \mathbb{R}^m$, the vector equation $x_1\mathbf{v}_1 + \ldots x_n\mathbf{v}_n = \mathbf{b}$ has a solution.
- The column vectors of A, namely $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ span $\mathbb{R}^m$.
- col $A = \mathbb{R}^m$.
- $\text{rank}(A) = m$
- The rows of A are linearly independent.
- The $m \times m$ matrix AA^T is invertible
- $AB = I$ for some $n \times m$ matrix B.

$$m = n \Updownarrow$$

- A is invertible.
- A can be reduced to I by a series of elementary row operations.
- $\det A \neq 0$.
- 0 is not an eigenvalue of A.
- The columns of A is a basis for $\mathbb{R}^n$.
- The rows of A is a basis for $\mathbb{R}^n$.

1 Last Words

- This is just a review sheet. Memorizing this review sheet won't guarantee that you will get a good mark in the final exam. Instead try to understand the logic and the proof behind each statement.
- There *will* be "proof" questions in the final exam. They will require some more logical thinking.
- If you are interested in a more "proof" settings and would like to know more of the theory, take MAT224. We will discuss more on general vector spaces, instead of $\mathbb{R}^n$; and change of bases, etc.
- Good luck in your final and happy studying.