MAT223 Topics and Tips

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Contents

Below is a list of topics considered "fair game" for the upcoming final exam. The list below just recaps a lot of what we've covered to help jog your memory and help you study. Notice that you may well be asked material from any point in the semester, although we will focus more heavily on material appearing since the first test. It's good to review every item in the list since **I will find ways to test you on your weakest topics**; you won't be able to sneak past our efforts to suss out where your knowledge is lacking so make sure you really master concepts.

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1 Topics

I've added a few things to the material so even if you read the topics on the list last time, give it another read since there have been a few updates on old topics.

1. **Systems of linear equations:** What they are, related vocabulary, geometrical interpretation. Solutions and consistency, *existence/uniqueness* etc. You should also be able to give solutions in a parametric vector form as in, for example, on page 45.

2. **Augmented matrices:** What they are, how to manipulate them via **elementary row operations**. This gave rise to the notion of **equivalence** between systems of linear equations.

3. **Reduction Algorithm:** using the reduction algorithm to produce the **general solution** to a given linear system. This meant learning about **row-echelon** and **reduced row-echelon** form for a matrix.

4. **Pivots:** We learned about pivot position, pivot columns, and pivots. You need to know what they are and how they were used. As well you need to know about **basic variables** and **free variables** and how they are used (as in building **parametric descriptions** of solution sets).

5. **Vectors:** You need to know about vectors in $\mathbb{R}^n$ i.e. what they are and the algebra of manipulating them.

6. **Vector Geometry:** You'll need to know about the **dot product** in $\mathbb{R}^n$ and related ideas like **length** of a vector and about the *angle between two vectors* in $\mathbb{R}^2$ as implicitly determined by the **dot product**. You also need to know about **orthogonality** and related concepts like the **Cauchy** and **Triangle** inequalities.

7. **Matrix-vector product:** If given an $m \times n$ matrix A and an $n \times 1$ vector $\mathbf{x}$ you should know how to compute $A\mathbf{x}$ – both in terms of taking a combination of the columns of A as well as using the dot product of the rows of A with $\mathbf{x}$. You'll need to know when the columns of A span $\mathbb{R}^m$ and how $A\mathbf{x} = \mathbf{b}$ is solved (if solvable).

8. **Homogeneous Systems:** You should know these are consistent. You should know words like **trivial solution** and **nontrivial solution** etc. You should know when they are guaranteed to have infinitely many solutions.

9. **Linear Combinations:** This is a fundamental recurring theme in this course and you need to know what it means.

10. **Spanning:** Given a collection of vectors you need to know what is meant by the **span** of the collection of vectors.

11. **Linear Independence:** You need to know what's meant by linear dependence and independence and how to tell whether a given collection of vectors is or is not linearly dependent/independent. Independence is, at heart, **a geometric concept**.

12. **Transformations:** You need to know what a transformation is and how to know *whether a given transformation is linear*, namely how this concept relates to the idea of linear combinations. We've seen that linear transformations are induced by associated matrix multiplications and vice versa. This is summarized in Theorem 10 on page 72 which you need to fully understand. You should be able to write down the **standard matrix** for a given linear transformation. Composition of linear transformations preserves the linearity and therefore results in a new linear transformation. Moreover, composition of linear transformations corresponds to multiplication of the associated matrices from which they are respectively induced. I.e., provided the sizes of A, B are compatible $T_A \circ T_B = T_{AB}$.

13. **Transformation jargon:** You need to know all the right terminology like "domain", "codomain", "onto" etc. You should be able to know how to answer questions about various properties of a transformation like determining if it's linear or whether it's onto etc.

14. **Matrix Arithmetic:** You need to know how to do scalar multiplication, addition, and the various arithmetical manipulations on matrices based on these. You need to know about size, and equality, and entries (*how to specify matrices given a description of the entries using indices*). You should know special matrices like the **zero matrix** and the **identity matrices**.

15. **Matrix Multiplication:** You need to know how to multiply matrices. This includes matrix-vector products, but also the more general case of compatible matrices. You should know the **dot product rule** for quickly multiplying matrices with compatible sizes. You should be able to look at matrices and tell if they're compatible. You should know the algebraic rules of matrices (associativity, lack of commutativity, etc) with respect to multiplication.

16. **Transposes:** You need to know about this, and associated concepts (like **symmetric matrices**). You should know how multiplication and the transpose operation interact with each other.

17. **Inverses:** You should know about matrix inverses, like the (incomplete list of) equivalent conditions on a matrix being invertible given on page 114. You should know how to find the inverse of an invertible matrix. As well, you should know how to use the **determinant** to find the inverse (and check invertibility) of a 2×2 matrix.

18. **Elementary Matrices:** You need to what they are, how to tell if a matrix is elementary, how to produce your own elementary matrices given a description of associated row operations, how to find the inverse of an elementary transformation etc.

19. **LU Factorization:** You need to know about the LU factorization. This amounts to knowing **how to perform the factorization** as well as **how to use the factored form** to solve a linear system. Are all matrices LU factorable?

20. **Spanning:** Given a collection of vectors you need to know what is meant by the span of the collection of vectors.

21. **Bases and dimension:** You need to know the definition of basis and how to determine if a given set is a basis. As well, you need to know how to use a basis to find the dimension of a subspace.

22. **Subspaces:** You need to know how to tell is a given subset is a subspace. There are canonical subspaces associated to a given matrix A called **null space** and **column**

space, denoted $Nul(A)$ and $col(A)$ respectively. You need to know about these subspaces and you should be able to characterize them, find bases for them and the like.

23. **Rank:** You need to know what rank is and the **rank-nullity theorem** described in the text as "The Rank Theorem" and in my notes as "Rank-nullity theorem". You need to know how rank extends the invertible matrix theorem as described on page 158 of the text say.

24. **More on Rank:** What can we say in cases where we have **full rank**? How are these cases related to the invertible matrix theorem?

25. **The Fundamental Theorem of Linear Algebra:** You need to be clear in your head what the theorem says[1] and what it indicates about solving linear systems. The statement of the theorem is in terms of **orthogonal complements** and so of course you must also be comfortable with orthogonal complements, how to prove basic facts with them, etc.

26. **Determinants:** You need to know how to calculate the determinant of an $n \times n$ matrix for any n. The best way to do this is via **cofactor expansion**, so you need to be comfortable with cofactors and how to use them. There are several *tricks* for speeding up the calculation of determinants, and you should make sure you get good at using them. As well, we've seen the **product theorem** which states that the determinant of a product of square matrices equals the product of the determinants of the matrices. That fact alone can often help derive new facts about the determinant.

27. **Eigenproblems:** Eigenvalue and eigenvector problems are ubiquitous in science so we expect you to have full facility in working with these types of problems. This includes the **characteristic polynomial** and how to use it. Finding eigenvectors should be a straightforward calculation for you to perform.

28. **Diagonalization:** If I give you a square matrix A can you find invertible P and diagonal D with $P^{-1}AP = D$? You should know some conditions which ensure this can be done (e.g. Theorem 7 on page 287) and if it can be done we expect you to be able to do it. That this very important topic appears at the end of the semester does **NOT** mean that it is somehow deemed less significant as far as the course is concerned. Quite the opposite, actually.

2 Concepts

Here are a few "big picture" items we've encountered this semester which are trees I hope don't got lost for the forest.

[1] And, not to state the obvious too much, if something is labelled *the Fundamental (___anything___)* that's obviously an important item.

1. **Subspaces:** These are *generalizations* of lines and planes through the origin. One has to check technical conditions to verify that something is or is not a subspace, but the basic idea is still to just extend the things which are hopefully very familiar to new territories.

2. **Geometry:** Gradually we went from considering systems of linear algebraic equations to describing geometric concepts (dimension, independence, orthogonality, etc). Somehow, moving away from thinking about coefficients and row reductions, and into the arena of matrix manipulations allowed us to start to describe things in a much richer, intuitive language. The deeper you go in studying linearity (like, say, MAT224) the more useful you'll find it to be able to use geometric intuition as a crutch in exploring new mathematical terrain.

3. **Unique solvability:** One recurrent theme is whether a system is *uniquely solvable*. For systems with as many equations as variables this is equivalent to asking if the associated coefficient matrix is invertible. We created a **long list of equivalent conditions** for a matrix being invertible. If I ask you what you can deduce from the fact that a given matrix is invertible, you ought to be able to *instantly* tell me about 8 things. If I gave you more time, you should be able to tell me plenty more....

4. **Arithmetic:** Matrix arithmetic simply **isn't the same** as arithmetic of numbers. Properties that you might *want* to hold, simply *don't* hold. So, in this way, matrices taunt your intuitions. But they also give you a glimpse at understanding and identifying the important arithmetical properties you may have been taking for granted. For every property about numbers you think you like, I can give you an example of a mathematical object which does not satisfy that property. Matrices are an example of a class of objects that meet *most* of the properties we like from numbers. More exotic mathematical objects fail in more mischievous (and interesting!) ways (though that's for other math courses).

5. **Claims require evidence:** If there's one thing that losing your intuitions (as above) should teach you, it's the importance of a clear and well-structured line of reasoning to separate fact from fiction. When your biases and intuitions are stripped away *the only reliable methodology* for determining the validity of a claim is formal, rigorous argumentation. On the final exam you may find yourself presented with a True/False type of claim, but the only way to *know* for sure involves demonstration of correctness rather that just a guess.

3 Takeaways

A few random thoughtlets about this course to keep in mind.

1. *All we have ever really done is try to look into solving systems of linear equations*. That's it. Admittedly, we developed complicated machinery to abstract the essential features of these systems and isolate the more salient questions/concerns, but

we have been focussed since day one on assigning values to variables where those variables are coming from solutions of systems of equations. Period.

2. *How you explain things matters*. **Our only way of determining what's in your head when you're thinking about a problem is what ends up on the test page,** so if you don't write your thoughts legibly and clearly (and using standard and precise notation) there's no way for us to be convinced that you actually know the material. Think of your exam booklet as your ambassador, representing your understanding of the concepts. You want your ambassador to represent you as well as possible. When we ask for you to "explain" something, we mean you should *really explain* it, in a way that accurately reflects the depth of your knowledge.

3. *Simple questions can often lead to complicated answers*. Often in science, asking very basic questions and pursuing a deep understanding of the answers can open all kinds of strange pathways. I would say that things like our rank theorems are the unexpected fruits from an innocent journey to understand linear systems. Determinants are like trolls under a bridge from coefficients to diagonalization. None of these things are easily anticipated. Such is mathematics...[2]

4. *Believe it or not, even the more abstract things we've covered this term have very important applications*. In your future science courses, particularly math, physics and statistics courses, you will see a lot more of the material we've covered this term. The abstraction wasn't abstraction for its own sake, it comes with benefits that serve a purpose and, beyond all math courses, this one may be the most important you take as an undergrad so don't think that we show you these things just to challenge your problem-solving skills. Linear algebra is a *way of thinking* that emerges over and over again in science and I hope you have learned to love it even a small bit as much as I do.

[2] Or, really, such is any science. Try asking a basic question in any field and pursue the answer and you'll see similar effects. If you don't believe me, try asking a physicist what things are made of. When they say atoms, ask what those are made of. When they say quarks and leptons interacting via mediating gauge bosons, ask what those things are. Keep pressing the question and watch the careful linguistic acrobatics.