Problem 1. (20 points total). This problem has six parts in total. Consider the following augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 2 & 3 & 1 \\ 2 & 2 & 4 & 2 \end{array}\right]$$

(i) (1 point). Write down the associated linear system for the above augmented matrix. **ANS:**

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 1 \\ 2x_1 + 2x_2 + 4x_3 &= 2 \end{aligned}$$

(ii) (8 points). Use elementary row operations to reduce the above system into reduced row echelon form. **ANS:**

$$\left[\begin{array}{ccc|c} 2 & 2 & 3 & 1 \\ 2 & 2 & 4 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

(iii) (3 points). Is the original system consistent or inconsistent? **ANS:** Consistent.

(iv) (6 points). Set $\mathbf{a}_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Is $\mathbf{b}$ a linear combination of $\mathbf{a}_1$ and $\mathbf{a_2}$? If not, explain. If yes, write $\mathbf{b}$ as a linear combination of $\mathbf{a}_1$ and $\mathbf{a_2}$. **ANS:** From part (ii) we have $x_1 = -1 - x_2$ and $x_3 = 1$, so, setting $x_2 = 0$ gives $-\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(v) (2 points). Is $\mathbf{b} \in \text{span}\{\mathbf{a}_1, \mathbf{a}_2\}$? Why or why not? **ANS:** Yes, because, as in the preceding, $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$.

Problem 2. (15 points total). This question has 2 parts.

(i) (10 points). Let $\mathbf{d} \neq \mathbf{0}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^n$. Use the Cauchy-Schwarz inequality to prove that $||\mathbf{proj_d v}|| \leq ||\mathbf{v}||$. **ANS:**

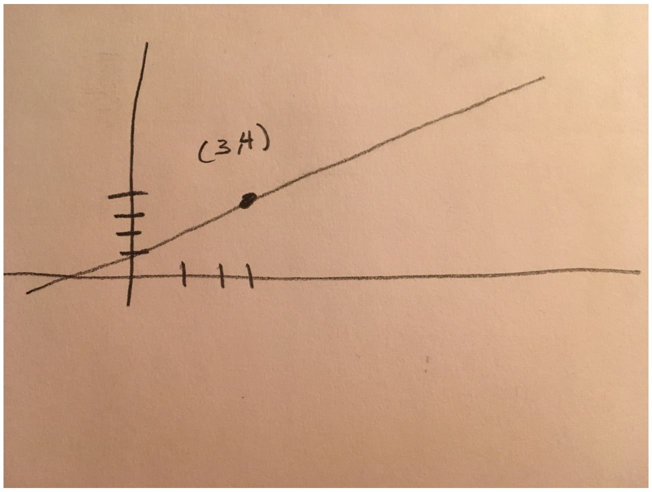
$$\begin{aligned} ||\mathbf{proj_d v}|| &= ||\frac{\mathbf{d} \cdot \mathbf{v}}{||d||^2}\mathbf{d}|| \\ &= (|\frac{\mathbf{d} \cdot \mathbf{v}}{||\mathbf{d}||^2}|)||\mathbf{d}|| \\ &\leq (\frac{||\mathbf{d}||||\mathbf{v}||}{||\mathbf{d}||^2})||\mathbf{d}|| \\ &= ||\mathbf{v}|| \end{aligned}$$

(ii) (5 points). When is the inequality in part (i) actually an equality? In other words, when is $||\mathbf{proj_d v}|| = ||\mathbf{v}||$? **ANS:** When the Cauchy inequality is an equality, namely, when $\mathbf{v} = c\mathbf{d}$ for some $c \in \mathbb{R}$.

Problem 3. (14 points total). This problem has 3 parts total. Consider two lines in $\mathbb{R}^2$ given by the parametric descriptions

$$L_1 = \{\begin{bmatrix}3\\4\end{bmatrix} + t\begin{bmatrix}1\\1\end{bmatrix} \mid t \in \mathbb{R}\} \quad \text{and} \quad L_2 = \{\begin{bmatrix}5\\4\end{bmatrix} + t\begin{bmatrix}1\\2\end{bmatrix} \mid t \in \mathbb{R}\}$$

(i) (3 points). Draw L_1. To receive full credit you must **clearly mark and label your axes** and draw a clear, accurate picture. **ANS:**



(ii) (4 points). Write down an equation of *any line* which intersects L_1 at a right angle. **ANS:** Can use any line in direction orthogonal to $\begin{bmatrix}1\\1\end{bmatrix}$ namely,

$$\{\begin{bmatrix}1\\-1\end{bmatrix} t \mid t \in \mathbb{R}\}$$

(iii) (7 points). Do L_1 and L_2 intersect? If so, find the point of intersection. **ANS:** Set $\begin{bmatrix}3\\4\end{bmatrix} + t\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}5\\4\end{bmatrix} + s\begin{bmatrix}1\\2\end{bmatrix}$ i.e. $t\begin{bmatrix}1\\1\end{bmatrix} + s\begin{bmatrix}-1\\-2\end{bmatrix} = \begin{bmatrix}2\\0\end{bmatrix} \implies \left[\begin{array}{cc|c}1 & -1 & 2\\1 & -2 & 0\end{array}\right] \sim \left[\begin{array}{cc|c}1 & 0 & 4\\0 & 1 & 2\end{array}\right]$ so $t = 4, s = 2$ i.e. **yes, the lines intersect** and the point of intersection is $(7, 8)$.

Problem 5. (21 points total, 3 points each) For the following questions, answer using the word "True" or the word "False". You **don't need to justify your answer** to receive full credit. There's no partial credit.

(i) True/False: If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ then $|\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}||^2\, ||\mathbf{v}||^2$. **ANS:** False. $u = 1$, $v = \frac{1}{2}$ fails.

(ii) True/False: If A is of size $n \times (n+1)$ and $\mathbf{x} \in \mathbb{R}^{n+1}$ then $A\mathbf{x} = \mathbf{0}$ is consistent. **ANS:** True. Homogeneous of any size is always consistent.

(iii) True/False: A linear system whose corresponding augmented matrix in reduced row echelon form has three pivots cannot be consistent. **ANS:** False. $x_1 = x_2 = x_3 = 0$ is a consistent system with three pivots in the RREF of the associated augmented matrix.

(iv) True/False: A linear system whose coefficient matrix has non-pivot columns must have infinitely many solutions. **ANS:** False. Consider $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$

(v) True/False: If $\mathbf{u} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ then $\operatorname{span}\{\mathbf{u}\} \subset \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ **ANS:** True, since $\mathbf{u} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \implies t\mathbf{u} \in \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for all $t \in \mathbb{R}$, and thus $\operatorname{span}\{\mathbf{u}\} = \{t\mathbf{u} \mid t \in \mathbb{R}\} \subset \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$

(vi) True/False: A homogeneous linear system may only have 1 or infinitely many solutions. **ANS:** True, since it cannot have 0 solutions.

(vii) True/False: If A is an $m \times n$ matrix whose columns do not span $\mathbb{R}^m$ then the equation $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b} \in \mathbb{R}^m$. **ANS:** True, this is discussed in section 1.4.