Homework #1

September 11, 2017

This is due at the beginning of class on **Tuesday, September 26**. Working in groups is fine but the write-up must be entirely your own and you should *list all collaborators* on the cover sheet of your submission.

1. Given a set Ω, we define the *power set* of Ω, $\mathcal{P}(\Omega)$ to be the set of all subsets of Ω. Prove that $\mathcal{P}(\Omega)$ is a σ-field of Ω.

2. Calculate the characteristic functions of the Cauchy distribution $C(\mu, \alpha)$ and the uniform distribution $\mathcal{U}(a, b)$.

3. Calculate the mean, variance and characteristic function of the exponential distribution $Exp(\theta, \mu) = \theta e^{-\theta(x-\mu)}$.

4. If $X \sim Poisson(\lambda)$ and $Y \sim Poisson(\theta)$ are independent Poisson-distributed random variables show that $X + Y \sim Poisson(\lambda + \theta)$

5. Suppose that $\mathbb{E}[|X|^n] < \infty$ for some $n \in \mathbb{N}$. Prove that the n'h derivative of the characteristic function of X exists and is continuous on $\mathbb{R}$ and that

   $$\phi_X^{(n)}(t) = \mathbb{E}[(iX)^n e^{itX}]$$

   and that the n'th moment of X around the origin is determined by

   $$\mathbb{E}[X^n] = \frac{1}{i^n} \phi_X^{(n)}(0)$$

6. Suppose that $X_0, X_1, X_2, ...$ is a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ and that $g : (\mathbb{R}, \mathcal{B}) \to (\mathbb{R}, \mathcal{B})$ is continuous. Prove that

   (a) If $X_n \xrightarrow{a.s.} X_0$ then $g(X_n) \xrightarrow{a.s.} g(X_0)$

   (b) If $X_n \xrightarrow{\mathbb{P}} X_0$ then $g(X_n) \xrightarrow{\mathbb{P}} g(X_0)$

   (c) If $X_n \xrightarrow{d} X_0$ then $g(X_n) \xrightarrow{d} g(X_0)$

7. Let X be an integer-valued random variable denoting the number of children of a certain kind of animal with $\mathbb{P}(X = 0) > 0$. We'll define X's *generating function* $g : [0, 1] \to [0, 1]$ to be determined by

   $$g(\theta) = \mathbb{E}[\theta^X], \qquad \theta[0, 1]$$

and we assume that $g'(1) = \lim_{\theta \uparrow 1} \frac{g(\theta)-g(1)}{\theta-1} < \infty$. Then we define

$$\{X_r^{(m)}\} = \#\{\text{children of the } r\text{'th animal in } m-1\text{'th generation}\}$$

Where each of the $X_r^{(m)}$ are independent and identically distributed random variables having the same distribution as X. The *size of the n'th generation* Z_n is then determined recursively via

$$Z_{n+1} = X_1^{(n+1)} + \cdots + X_{Z_n}^{(n+1)}, \qquad Z_0 = 1$$

We denote the generating function of Z_n by $s_n(\theta) = \mathbb{E}[\theta^{Z_n}]$ for $\theta \in [0, 1]$.

(a) Prove that $s_n = g \circ g \circ \cdots \circ g$, the n-fold iterated composition of the generating function of X. (*Hint: One way to do this requires using the fact about conditional expectations that* $\mathbb{E}[U] = \mathbb{E}[\mathbb{E}[U \mid V]]$)

(b) Defining $p_n = \mathbb{P}(Z_n = 0)$ as the chance of extinction by the n'th generation. Show that $p_n = g_n(0)$.

(c) Define the extinction probability $p = \mathbb{P}(\text{the animals will go extinct eventually})$. Use the above to prove that if $\mathbb{E}[X] > 1$ then p is the unique root of $x - g(x)$ and if $\mathbb{E}[x] \leq 1$ then $p = 1$.

8. Let $X \in \mathbb{R}^k$ be a random variable denoting the outcome of n independent rolls of a k-sided die (where p_i can denote the probability of side i facing up in a single toss). Each entry of X, namely X_i, should denote the number of times side i appeared in the n tosses. Find the covariance matrix X and prove that the characteristic function of X is

$$\phi_X(\boldsymbol{t}) = (\sum_{j=1}^{k} p_j e^{it_i})^n$$

9. (Hard) Recall the *Cauchy-Schwarz inequality* states that $|\mathbf{x} \cdot \mathbf{y}| \leq ||\mathbf{x}|| \cdot ||\mathbf{y}||$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Given a set of real numbers $S = \{x_1, ..., x_n\}$ we define the $\mu = \frac{1}{n} \sum_i x_i$ and $\sigma = \sqrt{\frac{1}{n} \sum_i (x_i - \mu)^2}$ to be the *mean* and *variance* of the numbers, respectively.

(a) Use the Cauchy-schwarz inequality to prove

$$(x_j - \mu)^2 \leq (n-1)\sigma^2$$

for all j.

(b) Next show that equality in the above holds if and only if $x_p = x_q$ holds for all $p, q \neq j$.

(c) Show the sharp bounds

$$\mu - \sigma\sqrt{n-1} \leq x_j \leq \mu + \sigma\sqrt{n-1}$$

and explain the meaning.