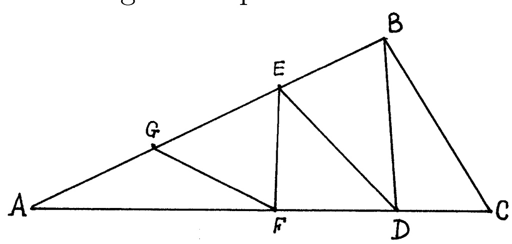
2 Math Club January 7, 2001

2.1 Warm-Up Problems

Problem 1. Triangle $\triangle ABC$ is given. Divide it by a broken line $BDEFG$ into five triangles of equal area.



Problem 2. Winnie Pooh and Piglet left their houses at the same time to visit one another. Due to the fog they did not notice one another. Piglet reached Winnie Pooh's place in four minutes and Winnie Pooh reached Piglet's place one minute after they "met". How long was everybody on his way?

2.2 How to solve problems?

2.2.1 Proof by contradiction

We want to prove some statement A. If assuming that the opposite statement is true we get a contradiction, then A is true.

Problem 1. Prove that there is no largest real number.

Problem 2. 5 children picked up together 9 mushrooms. Prove that at least two of them picked up the same number of mushrooms.

Problem 3. Prove that there is no tetrahedron such that every edge creates an obtuse angle with some other edge.

Problem 4. There are 100 numbers on a circle. Given that each number is equal to the arithmetical mean of its neighbours, prove that all these numbers are equal.

Problem 5. There are several points on the plane. It is given that any four of them create a convex quadrilateral. Prove that all of them create a convex polygon.

Problem 6. Prove that if n divides $(n-1)!+1$ then n is a prime.

Problem 7. Does there exist a convex polygon with more than 3 acute angles?

Problem 8. Prove that there is no polyhedron with an odd number of faces such that each face has an odd number of vertices.

2.3 Olympiads archive

Problem 1 (MathBattle) Point E is given on the diameter AC of the circle. Draw a chord BD through E in such a way that an area of the quadrilateral $ABCD$ is maximal.

Problem 2 (MathBattle) It is known that $m^2 + n^2 + m$ is divisible by mn for some positive integers m and n. Prove that m is a perfect square.

Problem 3 (Tournament of Towns) Let P be a point inside the triangle $\triangle ABC$ with $AB = BC$, $\angle ABC = 80°$, $\angle PAC = 40°$, and $\angle ACP = 30°$. Find $\angle BPC$.

Problem 4 (Tournament of Towns) The intelligence quotient (IQ) of a country is defined as the average IQ of its entire population. It is assumed that the total population and individual IQ's remain constant throughout.

(a) (i) A group of people from country A has emigrated to country B. Show that it can happen that as a result, the IQ's of both countries have increased.

(ii) After this, a group of people from B, which may include immigrants from A, emigrates to A. Can it happen that the IQ's of both countries will increase again?

(b) A group of people from country A has emigrated to country B, and a group of people from B has emigrated to country C. It is known that as a result, the IQ's of all three countries have increased. After this, a group of people from C emigrates to B and a group of people from B emigrates to A. Can it happen that the IQ's of all three countries will increase again?

Problem 5 (Tournament of Towns) You are given a balance and one copy of each ten weights of $1, 2, 4, 8, 16, 32, 64$, $128, 256$ and 512 grams. An object weighing M grams, where M is a positive integer, may be balanced in different ways by placing various combinations of the given weights on either pans of the balance.

(a) Prove that no object may be balanced in more than 89 ways.

(b) Find a value of M such that an object weighing M grams can be balanced in 89 ways.