

STA2111HF: Homework 4 Due Tuesday, Nov. 16

1. Let X, X_1, X_2, \dots be random variables with $X_n \rightarrow X$ a.s. and, for some $C < \infty$, for all n , $E[(X_n)^2] \leq C$. Show that $\lim_{n \rightarrow \infty} E[X_n] = E[X]$.
2.
 - i. Let X_1, X_2, \dots be independent with distributions μ_1, μ_2, \dots . Lyapunov's condition is that for some $\delta > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{s_n^{2+\delta}} \sum_{j=1}^n \int |x|^{2+\delta} d\mu_j = 0$$

- where $s_n^2 = \sigma_1^2 + \dots + \sigma_n^2$. Prove that in this situation, Lyapunov's condition is stronger than Lindeberg's.
- ii. X_1, X_2, \dots be independent with $X_j = \pm a_j$ each with probability $1/2$. What do Lindeberg's and Lyapunov's conditions require of the sequence a_j ? Is there a sequence which satisfies Lindeberg's condition, but does not satisfy Lyapunov's condition for any $\delta > 0$? Is there a sequence for which the CLT does not hold?
3. Show that if $X_n \rightarrow X$ in probability, then the associated distributions converge weakly.
 4. Let X_n be a sequence of random variables which is Cauchy in probability, meaning

$$\lim_{\substack{n \rightarrow \infty \\ m \rightarrow \infty}} P(|X_n - X_m| \geq \delta) = 0 \quad \text{for all } \delta > 0.$$

Show that there is a random variable X such that $X_n \rightarrow X$ in probability.

5. Give an example where the mean does not exist, but the weak law of large numbers still holds. (Hint. Try $f(x) = \frac{c}{(1+x^2)\log(1+x^2)}$).
6. Let X and Y be independent and f and g measurable functions. Prove or disprove: $f(X)$ and $g(Y)$ are independent.
7.
 - i. Show that if $\int |x| d\mu < \infty$ then the characteristic function φ is continuously differentiable and compute $\varphi'(0)$.
 - ii. Show that the converse is false: φ could be continuously differentiable but $\int |x| d\mu = \infty$. (Hint. Try something like $\mu(\{n\}) = \mu(\{-n\}) = \frac{1}{n^2 \log n}$.)
 - iii. Show that if $\int |x|^k d\mu < \infty$ then the characteristic function φ is k times continuously differentiable and compute the k th derivative at 0.
 - iv. Show that the converse is true for even k .
 - v. Show that e^{-t^4} is not a characteristic function.
8. Show that the covariance matrix

$$C_{ij} = \text{Cov}(X_i, X_j)$$

of X_1, \dots, X_n is a symmetric positive (semi)definite matrix (look it up if you don't know the definition, e.g. on wikipedia). Is every such matrix the covariance matrix of a random vector?

9. Show that

$$d(F, G) = \inf_{\epsilon > 0} \{F(x - \epsilon) - \epsilon \leq G(x) \leq F(x + \epsilon) + \epsilon \text{ for all } x\}$$

defines a metric on the space of distributions (again, if you don't know the definition, look it up on wikipedia). It is called the Lévy metric. Draw a distribution function with at least one discontinuity and a picture of the unit ball around that df. Show that convergence in this metric is equivalent to weak convergence.

10. Let μ_n , $n = 1, 2, \dots$ be Gaussian measures on the real line with means m_n and variances σ_n^2 . Find a necessary and sufficient condition on the m_n and σ_n so that the sequence is tight.
11. Let X_1, X_2, \dots be independent and uniformly bounded. Suppose that $\sum_{i=1}^{\infty} \text{Var}(X_i) = \infty$ and $S_n = X_1 + \dots + X_n$. Show that

$$\frac{S_n - E[S_n]}{\sqrt{\text{Var}(S_n)}}$$

has an asymptotically Gaussian distribution.