STA2111HF: Homework 5 Due Tuesday, Nov. 30

1. Let $X_1, X_2, \dots$ be independent identically distributed random variables with mean 0 and variance 1.
   i. Show that $Y = \limsup_{n\to\infty} \frac{X_1+\cdots+X_n}{\sqrt{n}}$ is a tail event.
   ii. Use the CLT to show that $P(Y \geq x) > 0$ for any x.
   iii. Use the Kolmogorov $0-1$ law to conclude that $Y = \infty$ almost surely.
   iv. Compute all possible limit points of the sequence $\left\{\frac{X_1+\cdots+X_n}{\sqrt{n}}\right\}_{n=1,2,\dots}$
2. In each of the following, μ is an infinitely divisible probability measure on $\mathbf{R}$.
   i. Suppose that μ is supported on a finite set, i.e. there exist $x_1, \dots, x_n$ such that $\sum_{i=1}^n \mu(\{x_i\}) = 1$. Find μ.
   ii. Suppose that μ is supported on a bounded set, i.e. there exists $K > 0$ with $\mu([-K, K]) = 1$. Find μ.
   iii. Let $\varphi(t) = \int e^{itx} d\mu(x)$. Find all solutions of $\varphi(t) = 0$ (and prove it. Hint: $|\varphi|^2$ has the same zero's.)
3. Show that the product of infinitely divisible characteristic functions is infinitely divisible. Show that the pointwise limit of infinitely divisible characteristic functions is infinitely divisible.
4. Suppose that μ_n are a sequence of probability measures on $\mathbf{R}$ with densities f_n satisfying $\sup_n \int_{-\infty}^{\infty} f_n^2(x)dx < \infty$. Prove or disprove: μ_n are tight.
5. Suppose that $X_1, X_2, \dots$ is a sequence of independent identically distributed random variables and there exists a random variable Y such that

$$\lim_{n\to\infty} \frac{X_1+\cdots+X_n}{n} = Y \qquad \text{a.s.}$$

   Show that $E[|X_1|] < \infty$ and $Y = E[X_1]$ a.s. (Hint: Show that $E[X] - 1 \leq \sum_n P(X \geq n) \leq E[X]$ and hence that $E[X] < \infty$ if and only if $\sum_n P(X \geq n) < \infty$.)
6. Let $X_1, X_2, \dots$ be a sequence of independent random variables. Show that $\sum_{n=1}^\infty X_n^2$ converges almost surely if and only if $\sum_{n=1}^\infty E\left[\frac{X_n^2}{1+X_n^2}\right]$ converges.
7. Let $X_1, X_2, \dots$ be a sequence of independent identically distributed random variables.
   i. If the X_i are Gaussian with mean 0 and variance 1, compute $\limsup_n \frac{X_n}{\sqrt{\log n}}$.
   ii. If the X_i are Poisson with parameter $\lambda > 0$ compute $\limsup_n \frac{X_n \log\log n}{\log n}$.
   iii. If the X_i have symmetric stable law with parameter $\alpha \in (0, 2)$, and $S_n = X_1 + \cdots + X_n$, compute $\limsup_n \left|\frac{S_n}{n^{1/\alpha}}\right|^{\frac{1}{\log\log n}}$.
8. (Corrected from HW4) The Lévy metric is

$$d(F, G) = \inf_{\epsilon>0}\{F(x-\epsilon) - \epsilon \leq G(x) \leq F(x+\epsilon) + \epsilon \text{ for all } x\}$$

   Draw a distribution function with at least one discontinuity and a picture of the ball of radius 1/100 around it.

9. If μ and ν are probability measures on $(\Omega, \mathcal{F})$ then the total variation distance between μ and ν is

$$\|\mu - \nu\|_{TV} = 2 \sup_{A \in \mathcal{F}} |\mu(A) - \nu(A)|$$

i. Show that this defines a metric on the space of probability measures. Is there any relation between this kind of convergence and weak convergence?
ii. Show that $\|\mu - \nu\|_{TV} = \sup_{|g| \leq 1} |\int g d\mu - \int g d\nu|$ where the supremum is taken over measurable g.
iii. If μ and ν have densities f and g, compute the total variation in terms of them.
iv. If μ and ν are discrete measures (supported on finite sets) compute the total variation in terms of the probabilities of the finitely many values μ and ν take.
v. Show that

$$\|\mu_1 \otimes \mu_2 - \nu_1 \otimes \nu_2\|_{TV} \leq \|\mu_1 - \nu_1\|_{TV} + \|\mu_2 - \nu_2\|_{TV}$$

where $\mu \otimes \nu$ denotes the product measure.
vi. Let μ and ν be discrete measures. Show that $\|\mu - \nu\|_{TV} \leq \epsilon$ if and only if there exist random variables $X \sim \mu$ and $Y \sim \nu$ with $P(X \neq Y) \leq \epsilon/2$.