STA2111HF: Practice final

1. Show that if a sequence of characteristic functions converges to 1 in an interval around 0, then it converges to 1 for all real t (Hint: Show that $1 - \cos 2t \leq 4(1 - \cos t)$ NOTE TYPO IN HANDOUT VERSION)

2. Let X and Y be real valued random variables and assume that $X \in \sigma(Y)$. Show that there exists a Borel measurable $g : R \to R$ so that $X = g(Y)$. Hint: first try the case when X is a simple function.

3. Let S_n denote the position of a simple random walk after n steps. (i.e. $S_n = X_1 + \cdots + X_n$ with X_i i.i.d. $P(X_i = \pm 1) = 1/2$.) For which polynomials $g(x)$ will $g(S_n)$ be a martingale?

4. Let $X_1, X_2, \ldots$ be independent random variables and $S_n = X_1 + \cdots + X_n$. Assume that almost surely $|X_i| \leq M$ for all $i \geq 1$ with a given constant $M < \infty$. Show that if $Var(S) \to \infty$ then $\frac{S_n - E[S_n]}{Var^{1/2}(S_n)} \Rightarrow N(0, 1)$.

5. Let $X_1, X_2, \ldots$ be i.i.d. standard normals. Find a deterministic sequence a_n so that
$$Y_n = a_n \frac{X_1}{\sqrt{X_1^2 + \cdots + X_n^2}}$$
converges weakly to a non-constant distribution and identify the limit.

6. Prove or disprove: The difference of two identically distributed independent random variables can be uniformly distributed on $[-1, 1]$.

7. Prove or disprove: If X_n converges in distribution to a constant c then the convergence also holds in probability.

8. Let $X_1, X_2, \ldots$ i.i.d with distribution function $F(x)$. Denote the maximum of the first n element by M_n. Show that if $\lim_{x \to \infty} x^a (1 - F(x)) = b$ for some positive constants a and b then $n^{-1/a} M_n$ converges in distribution and identify the limiting distribution. Hint: you can use the convergence of the distribution functions to prove the weak limit.

9. Let $X_1, X_2, \ldots$ be independent with the following distribution: $P(X_m = m) = P(X_m = -m) = \frac{1}{2m^2}$, $P(X_m = 1) = P(X_m = -1) = \frac{1}{2} - \frac{1}{2m^2}$. Let $S_n = X_1 + \cdots + X_n$. Show that $Var(S_n) \to 2$,but $S_n \Rightarrow N(0,1)$. Why doesn't this contradict the central limit theorem?

10. Let $X_1, X_2, \ldots$ be i.i.d. with exponential distribution with parameter λ. Compute $\limsup_{n\to\infty} X_n / \log n$.

11. Find a sequence of independent, non-negative random variables with mean one such that

$$\limsup_{n\to\infty} \frac{X_1 + \cdots + X_n}{n} = \infty \quad a.s.$$

(Hint: They cannot be identically distributed. Why?)

12. Let $X_1, X_2, \ldots$ be independent random variables with $P(X_n = n) = P(X_n = -n) = \frac{1}{2n\log(n+1)}$, and with the rest of the probability, $X_n = 0$. Show that $\frac{X_1 + \cdots + X_n}{n}$ converges to 0 in probability, but it almost surely does not converge to a finite limit.