Conditional Expectation

Probability space $(\Omega, \mathcal{F}, P)$
Random variable $X \in L^1$
Sub $\sigma-$field $\mathcal{G} \subset \mathcal{F}$

Definition

The conditional expectation of X given $\mathcal{G}$ is a random variable $E[X|\mathcal{G}]$ satisfying

1. $E[X|\mathcal{G}] \in \mathcal{G}$
2. $\int_A XdP = \int_A E[X|\mathcal{G}]dP$ for all $A \in \mathcal{G}$

$X \geq 0$, $Q(A) = \int_A XdP$, $A \in \mathcal{G}$. Q measure on $(\Omega, \mathcal{G}, P)$, $Q << P$

Radon-Nikodym theorem: $\exists \frac{dQ}{dP} \in L^1(\Omega, \mathcal{G}, P)$ s.t. $Q(A) = \int_A \frac{dQ}{dP} dP$

$E[X|\mathcal{G}] = \frac{dQ}{dP}$

if $X = X_+ - X_-$ define $E[X|\mathcal{G}] = E[X_+|\mathcal{G}] - E[X_-|\mathcal{G}]$

Examples.

1. $\mathcal{G} = \{\emptyset, \Omega\}$, $E[X|\mathcal{G}] = E[X]$
2. $\mathcal{G} = \mathcal{F}$, $E[X|\mathcal{G}] = X$
3. $\Omega = [0, 1)$, $\mathcal{F} =$Borel sets, $P =$ Lebesgue, $\mathcal{F}_n =$ Dyadic level n, $E[X|\mathcal{F}_n](\omega) = Av_{[\frac{i}{2^n}, \frac{i+1}{2^n})}X$, $\omega \in [\frac{i}{2^n}, \frac{i+1}{2^n})$
4. $A_1, A_2, \ldots$ partition of Ω. $\mathcal{G}$ is $\sigma-$field generated by this partition $E[X|\mathcal{G}] = \frac{1}{P(A_i)} \int_{A_i} XdP$, $\omega \in A_i$

   In particular if $A_1 = A$, $A_2 = A^C$, $X = 1_B$ then $E[1_B|\mathcal{G}] = P(B|A) = \frac{P(B \cap A)}{P(A)}$ on A
5. $P((X, Y) \in A) = \int_A f(x, y)dxdy$
   $E[g(X)|Y] = \frac{\int g(x)f(x,y)dx}{\int f(x,y)dx} = \int g(x)P(X \in dx|Y = y)$
6. $X \in G \Rightarrow E[XY|\mathcal{G}] = XE[Y|\mathcal{G}]$
7. X indep of $\mathcal{G} \Leftrightarrow E[X|\mathcal{G}] = E[X]$

Martingales: Discrete time

Definition.

An non-decreasing family of sub-σ-fields $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \mathcal{F}$ is called a *filtration*

Definition

M_n a sequence of random variables in $L^1(\Omega, \mathcal{F}, P)$. If

$$E[M_{n+1} \mid \mathcal{F}_n] = M_n$$

then M_n is a *martingale* with respect to the filtration $\mathcal{F}_n$

submartingale: $E[M_{n+1} \mid \mathcal{F}_n] \geq M_n$

supermartingale: $E[M_{n+1} \mid \mathcal{F}_n] \leq M_n$

Example. $S_n = X_1 + \cdots + X_n$, X_i iid

$E[X_i] = 0 \Rightarrow S_n$ martingale. $E[X_i] \geq 0 \Rightarrow S_n$ submartingale.
$E[X_i] \leq 0 \Rightarrow S_n$ supermartingale

Lemma

Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex and X_n a martingale with respect to $\mathcal{F}_n$. Then $\phi(X_n)$ is a submartingale with respect to $\mathcal{F}_n$.

Proof.

By Jensen's inequality for conditional probability

$$E[\phi(X_n) \mid \mathcal{F}_n] \geq \phi(E[X_n \mid \mathcal{F}_n]) = \phi(X_n).$$

□

Example. $S_n = X_1 + \cdots + X_n$, X_i iid, $E[X_i] = 0$, $Var(X_i) = \sigma^2 < \infty$

S_n martingale. S_n^2 submartingale. $S_n^2 - \sigma^2 n$ martingale.

$E[S_{n+1}^2 - \sigma^2(n+1)|\mathcal{F}_n] = E[S_n^2 + 2S_nX_{n+1} + X_{n+1}^2 - \sigma^2(n+1)|\mathcal{F}_n] = S_n^2 + \sigma^2 - \sigma^2(n+1)$

Doob's inequality (Discrete time)

Let X_n be a submartingale with respect to $\mathcal{F}_n$. Then for any $\lambda > 0$ and $n = 1, 2, \ldots$,

$$P\left(\max_{1\le k\le n} X_k \ge \lambda\right) \le \frac{E[X_n^+]}{\lambda}.$$

Proof.

$A_i = \left\{X_i \ge \lambda, \max_{0\le k\le i-1} X_k < \lambda\right\}$ disjt $\cup_{i=1}^n A_i = \{\max_{1\le i\le n} X_i \ge \lambda\}$

$$P\left(\max_{1\le i\le n} X_i \ge \lambda\right) = \sum_{i=1}^n P(A_i) \le \sum_{i=1}^n \frac{1}{\lambda}\int_{A_i} X_i dP \qquad \textit{Tchebyshev}$$

$$\le \sum_{i=1}^n \frac{1}{\lambda}\int_{A_i} E[X_n \mid \mathcal{F}_i] dP = \sum_{i=1}^n \frac{1}{\lambda}\int_{A_i} X_n dP$$

$$= \frac{1}{\lambda}\int_{\{\max_{1\le i\le n} X_i \ge \lambda\}} X_n dP$$

Example

$B'(t)$ is formally White noise so formally $B'(t) = \sum_n Z_n e^{2\pi int}$, Z_n iid $\mathcal{N}(0,1)$ so we expect $B(t) = \sum_n Z_n \frac{e^{2\pi int}-1}{2\pi in}$ Does it converge?

Kolmogorov Three Series Theorem

$X_1, X_2, \ldots$ independent. $\sum_{n=1}^{\infty} X_n$ converges if and only if (1) $\sum_{n=1}^{\infty} P(|X_n| > M) < \infty$;(2) $\sum_{n=1}^{\infty} E[X_n^M] < \infty$;(3) $\sum_{n=1}^{\infty} Var(X_n^M) < \infty$, for all $M > 0$ where $X_n^M = X_n 1_{|X_n| \leq M}$.

Proof of "if"

Let $\bar{X}_n^M = X_n^M - E[X_n^M]$. By Doob's inequality,

$$P(\max_{N \leq m \leq R} |\sum_{n=N+1}^{m} \bar{X}_n^M| \geq \epsilon) \leq \epsilon^{-2} \sum_{n=N+1}^{R} Var(X_n^M)$$

By (3) rhs$\downarrow 0$ as $N \uparrow \infty$ uniformly in R, so $\sum_{n=1}^{N} \bar{X}_n^M$ is Cauchy, hence convergent. Now (2) $\Rightarrow \sum_{n=1}^{N} X_n^M$ convergent
(1)+Borel-Cantelli $\Rightarrow X_n^M = X_n$ except for finitely many n. Q.E.D.

Definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_n$, $n = 0, 1, 2, \ldots$ a filtration A random variable τ taking values in $\{0, 1, 2, \ldots\}$ is called a *stopping time* if for each $n = 0, 1, 2, \ldots$,

$$\{\omega \in \Omega \ : \ \tau(\omega) \leq n\} \in \mathcal{F}_n.$$

Example

Let X_n be a random walk starting at 0. Let $\tau = \min\{n \geq 0 \ : \ X_n \geq a\}$ be the *first passage time* of level a. τ is a stopping time.

Let $\sigma = \max\{n \geq 0 \ : \ X_n \leq a\}$, the *last passage time*. σ is not a stopping time.

$$\mathcal{F}_\tau = \{A \in \mathcal{F} \ : \ A \cap \{\tau \leq n\} \in \mathcal{F}_n, \ n \geq 0\}$$

is a σ-field representing the information up to the stopping time τ

Optional stopping

M_n martingale wrt filtration $\mathcal{F}_n$. $\tau \geq \sigma$ bounded stopping times

$$E[X_\tau \mid \mathcal{F}_\sigma] = X_\tau$$

bounded means $\tau \leq B$ Otherwise it is FALSE

Proof.

Need: $\int_A X_\tau dP = \int_A X_\sigma dP, \quad \forall A \in \mathcal{F}_\sigma$
$\int_{A\cap\{\sigma=\ell\}} X_\sigma dP = \int_{A\cap\{\sigma=\ell\}} X_B dP$ since $A \cap \{\sigma = \ell\} \in \mathcal{F}_\ell$
so $\int_A X_\sigma dP = \int_A X_B dP$ same for X_τ since $\mathcal{F}_\sigma \subset \mathcal{F}_\tau$

□

Example

$X_1, X_2, \ldots$ iid $P(X_i = 1) = P(X_i = -1) = 1/2$
$S_n = X_1 + \cdots + X_n$ Random walk
$\tau_{\pm a} = \min\{n : |S_n| = a\} \quad E[\tau_{\pm a}] = ?$
$S_n^2 - n$ martingale
$\tau_{\pm a}^B = \min\{\tau_{\pm a}, B\}$ bounded stopping time
Optional stopping: $E[S^2_{\tau^B_{\pm a}} - \tau^B_{\pm a}] = 0$
$\lim_{B\uparrow\infty} E[\tau^B_{\pm a}] = E[\tau_{\pm a}]$ by monotone convergence theorem
$\lim_{B\uparrow\infty} E[S^2_{\tau^B_{\pm a}}] = a^2$ by bounded convergence theorem
$E[\tau_{\pm a}] = a^2$

Counterexample

Try same for $\tau_a = \min\{n : S_n = a\}$
$\lim_{B\uparrow\infty} E[S^2_{\tau^B_a}] = E[\tau_a]$
but $\lim_{B\uparrow\infty} E[S^2_{\tau^B_a}] = \infty \neq E[S^2_{\tau_a}] = a^2$

Martingales: Continuous time

Definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space

$\mathcal{F}_t$, $t \geq 0$ a *filtration* ($=$ non-decreasing family of sub-σ-fields of $\mathcal{F}$)

M_t, $t \geq 0 \in L^1$ is a *martingale* with respect to $\mathcal{F}_t$, $t \geq 0$ if whenever $s \leq t$,

$$E[M_t \mid \mathcal{F}_s] = M_s.$$

submartingale if $\geq$ *supermartingale* if $\leq$

Examples

- B_t is a martingale wrt $\mathcal{F}_t = \sigma(B_s, s \leq t)$
- B_t^2 is a submartingale
- $B_t^2 - t$ is a martingale
- $e^{\lambda B_t - \frac{1}{2}\lambda^2 t}$ is a martingale for any $\lambda \in \mathbb{R}$

Martingale characterization of Brownian motion

If $e^{\lambda B_t - \frac{1}{2}\lambda^2 t}$ is a martingale wrt $\mathcal{F}_t = \sigma(B_s, s \leq t)$ for any $\lambda \in \mathbb{R}$ then B_t, $t \geq 0$ is Brownian motion

Proof.

$$E[e^{\lambda(B_t - B_s)} | \mathcal{F}_s] = e^{\frac{1}{2}\lambda^2 (t-s)}$$

so $B_t - B_s$ independent of $\mathcal{F}_s$ and $\mathcal{N}(0, t-s)$

□

Doob's inequality

If X_t is a submartingale with respect to $\mathcal{F}_t$ and the paths of X_t are right continuous with probability one, then

$$P(\sup_{0\leq t\leq T} X_t \geq \lambda) \leq \frac{E[X_T^+]}{\lambda}$$

Proof.

Let $0 \leq t_0 < t_1 < \cdots$ $\tilde{X}_n = X_{t_n}$ is a martingale wrt $\tilde{\mathcal{F}}_n = \mathcal{F}_{t_n}$.

$$P(\sup_{0\leq t_i\leq T} X_{t_i} \geq \lambda) \leq \frac{E[X_T^+]}{\lambda}$$

By right continuity lhs $\uparrow P(\sup_{0\leq t\leq T} X_t \geq \lambda)$ as mesh$\downarrow 0$ □

Optional stopping

X_t, $t \geq 0$ be a right continuous martingale with respect to $\mathcal{F}_t$, $t \geq 0$ and $\sigma \leq \tau$ bounded stopping times

$$E[X_\tau \mid \mathcal{F}_\sigma] = X_\sigma$$

Proof.

$\sigma_n = 2^{-n}(\lfloor 2^n \sigma \rfloor + 1)$
$\tau_n = 2^{-n}(\lfloor 2^n \tau \rfloor + 1)$
$\sigma_n \leq \tau_n \leq B$
$E[X_{\tau_n} \mid \mathcal{F}_{\sigma_n}] = X_{\sigma_n}$ ie $\int_A X_{\tau_n} dP = \int_A X_{\sigma_n} dP$, $A \in \mathcal{F}_\sigma$,since $\sigma \leq \sigma_n$
By right continuity $X_{\tau_n} \to X_\tau$ and $X_{\sigma_n} \to X_\sigma$

Recall $\{X_n\}_{n=1,2,\ldots}$ is *uniformly integrable* if

$$\lim_{M \uparrow \infty} \sup_n \int_{|X_n| \geq M} |X_n| dP = 0$$

and if $X_n \xrightarrow{a.s.} X$ then $\{X_n\}_{n=1,2,\ldots}$ uniformly integrable $\Leftrightarrow X_n \xrightarrow{L^1} X$

X_{σ_n}, $n = 1, 2, \ldots$ and X_{τ_n}, $n = 1, 2, \ldots$ are *backwards martingales* with respect to $\mathcal{F}_n$, $n = 1, 2, \ldots$, i.e. $E[X_{\sigma_{n-1}} \mid \mathcal{F}_{\sigma_n}] = X_{\sigma_n}$

Lemma

A backwards martingale is uniformly integrable

Proof.

$E[X_m \mid \mathcal{F}_n] = X_n$ whenever $m \leq n$ so $|X_n| \leq E[|X_0| \mid \mathcal{F}_n]$
so

$$\int_{\{|X_n|>\ell\}} |X_n| dP \leq \int_{\{|X_n|>\ell\}} |X_0| dP = \int 1_{\{|X_n|>\ell\}} |X_0| dP$$

$$P(|X_n| > \ell) \leq \frac{E[|X_n|]}{\ell} \leq \frac{E[|X_0|]}{\ell}$$

so $1_{\{|X_n|>\ell\}} |X_0| \stackrel{a.s.}{\to} 0$

$\int_{\{|X_n|>\ell\}} |X_0| dP \to 0$ by dominated convergence theorem □