## Feynman-Kac formula

$V$ a nice function (say bounded). $u \in C^{1,2}$ solves

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + Vu, \qquad u(0,x) = u_0(x)$$

$\int u_0(x)\exp\{-x^2/2t\}dx < \infty$. Then

$$u(t,x) = E_x[e^{\int_0^t V(B(s))ds} u_0(B(t))]$$

## Proof.

For $0 \le s \le t$ let $Z(s) = u(t-s, B(s))e^{\int_0^s V(B(u))du}$. By Itô's lemma

$$\begin{aligned} Z(t) - Z(0) &= \int_0^t \left\{-\frac{\partial u}{\partial s} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + Vu\right\} e^{\int_0^s V(B(u))du} ds \qquad = 0 \\ &\quad + \int_0^t \frac{\partial u}{\partial x}(t-s, B(s)) e^{\int_0^s V(B(u))du} dB(s) \qquad = \text{mart} \end{aligned}$$

$$\Rightarrow \quad E_x[Z(t)] = E_x[Z(0)]$$

□

1. In $d > 1$ if $u \in C^{1,2}$ solves
$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + Vu, \quad u(0,x) = u_0(x)$$
then
$$u(t,x) = E_x\left[e^{\int_0^t V(B(s))ds}u_0(B(t))\right]$$
where $B(t)$ is $d$-dimensional Brownian motion
2. If $V = V(t,x)$
$$u(t,x) = E_x\left[e^{\int_0^t V(t-s,B(s))ds}u_0(B(t))\right]$$
3. Feynman: solution $u$ of it Schrödinger equation $\frac{\partial u}{\partial t} = i[\frac{1}{2}\frac{\partial^2 u}{\partial x^2} + Vu]$ has "representation"
$$u(x,t) = \int_{f:f(0)=x} e^{i\int_0^t V(f_s)ds - \frac{i}{2}\int_0^t |f'|^2 ds}u_0(f_t)d\mu$$
where $\mu$ is translation invariant measure on space of functions
BUT, No such measure $\mu$
Kac pointed out that it is rigorous if $i \mapsto 1$ because
$e^{-\frac{1}{2}\int_0^t |f'|^2 ds}d\mu \overset{formally}{=}$ Brownian motion

# The Monte Carlo Method

Problem. Given $V$ compute

$$\lambda_0 = \sup \mathrm{spectrum}(\frac{1}{2}\Delta + V)$$

Idea (Ulam, Fermi, von Neumann, Metropolis)

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + Vu, \quad u(0,x) = 1$$

$$\lambda_0 = \lim_{t\to\infty} \frac{1}{t} \log u(0,t) = \lim_{t\to\infty} \frac{1}{t} \log E_0[e^{\int_0^t V(B(s))ds}]$$

Simulate $N$ Brownian paths $B_1, \ldots B_N$, $N$ large
Take $t$ large

$$\lambda_0 \approx \frac{1}{t} \log \frac{1}{N} \sum_{i=1}^{N} e^{\int_0^t V(B_i(s))ds}$$

## Arcsin law

$\xi(t) = \frac{1}{t}\int_0^t \mathbf{1}_{[0,\infty)}(B(s))ds$ = the fraction of time that Brownian motion is positive up to time $t$

$$P(\xi(t) \le a) = \begin{cases} 0 & a < 0; \\ \frac{2}{\pi}\arcsin\sqrt{a} & 0 \le a \le 1; \\ 1 & a > 1. \end{cases}$$

Simple explanation why distribution of $\xi(t)$ indep of $t$

$$\xi(t) = \int_0^1 \mathbf{1}_{[0,\infty)}(B(ts))ds = \int_0^1 \mathbf{1}_{[0,\infty)}(\frac{1}{\sqrt{t}}B(ts))ds = \int_0^1 \mathbf{1}_{[0,\infty)}(\tilde{B}(s))ds$$

$$\xi(t) \stackrel{d}{=} \xi(1)$$

## Proof

By Feynman-Kac if we can find a nice solution of

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - u\mathbf{1}_{[0,\infty)} \qquad\qquad u(0,x) = 1$$

Then

$$u(t,x) = E_x[e^{-\int_0^t \mathbf{1}_{[0,\infty)}(B(s))ds}]$$

and

$$u(t,0) = \int_0^1 e^{-at} dP(\xi \le a)$$

$$\alpha > 0,\ \phi_\alpha(x) = \alpha\int_0^\infty u(t,x)e^{-\alpha t}dt \ \longrightarrow\ -\tfrac{1}{2}\phi_\alpha'' + (\alpha + \mathbf{1}_{[0,\infty)})\phi_\alpha = \alpha$$

$$\phi_\alpha(x) = \begin{cases} \frac{\alpha}{\alpha+1} + Ae^{x\sqrt{2(\alpha+1)}} + Be^{-x\sqrt{2(\alpha+1)}}, & x \ge 0, \\ 1 + Ce^{x\sqrt{2\alpha}} + De^{-x\sqrt{2\alpha}}, & x \le 0. \end{cases}$$

$$u \le 1 \Rightarrow \phi_\alpha \le 1 \Rightarrow A = D = 0$$

## Proof.

$\phi_\alpha(0_-) = \phi_\alpha(0_+), \phi'_\alpha(0_-) = \phi'_\alpha(0_+)$

$\Rightarrow B = \frac{\alpha^{1/2}}{(1+\alpha)(\sqrt{\alpha}+\sqrt{\alpha+1})}, C = \frac{1}{(1+\alpha)^{1/2}(\sqrt{\alpha}+\sqrt{\alpha+1})^{1/2}}$

$$\phi_\alpha(0) = \sqrt{\frac{\alpha}{\alpha+1}} = \int_0^\infty E[e^{-t\xi}\alpha e^{-\alpha t}]dt$$

By Fubini's theorem this reads $E\left[\frac{\alpha}{\alpha+\xi}\right] = \sqrt{\frac{\alpha}{\alpha+1}}$ or

$$\int_0^1 \frac{1}{1+\gamma a} dP(\xi \leq a) = \frac{1}{\sqrt{1+\gamma}}$$

Looking up a table of transforms we find

$$dP(\xi \leq a) = \frac{2}{\pi}\frac{1}{\sqrt{a(1-a)}}da \qquad 0 \leq a \leq 1$$

which is the density of the arcsin distribution □

# Stochastic differential equations

$\sigma(x, t)$, $b(x, t)$ mble

## Definition

A stochastic process $X_t$ is a solution of a stochastic differential equation

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t, \qquad X_0 = x_0$$

on $[0, T]$ if $X_t$ is progressively measurable with respect to $\mathcal{F}_t$, $\int_0^T |b(X_t, t)|dt < \infty$, $\int_0^T |\sigma(X_t, t)|^2 dt < \infty$ a.s. and

$$X_t = x_0 + \int_0^t b(X_s, s)ds + \int_0^T \sigma(X_s, s)dB_s \qquad 0 \le t \le T$$

The main point is that $\sigma(\omega, t) = \sigma(X_t, t)$, $b(\omega, t) = b(X_t, t)$
Under reasonable conditions the solution $X_t$ exists, is unique, and is a Markov process

# Ornstein-Uhlenbeck Process

$X_t$, $t \geq 0$ is the solution of the *Langevin equation*

$$dX_t = -\alpha X_t dt + \sigma dB_t, \qquad \alpha > 0$$

To solve it

$$de^{\alpha t} X_t = \alpha e^{\alpha t} X_t dt + e^{\alpha t}(-\alpha X_t dt + \sigma dB_t) = \sigma e^{\alpha t} dB_t$$

so

$$X_t = X_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dB_s$$

If $X_0 \sim \mathcal{N}(m, V)$ indep of $B_t$, $t \geq 0 \Rightarrow X_t$ Gaussian process

$$m(t) = E[X_t] = me^{-\alpha t}$$

$$c(s,t) = \mathrm{Cov}(X_s, X_t) = [V + \frac{\sigma^2}{2\alpha}(e^{2\alpha \min(t,s)} - 1)]e^{-\alpha(t+s)}$$

$m = 0$, $V = \frac{\sigma^2}{2\alpha} \Rightarrow X_t$ stationary Gaussian $c(s,t) = \frac{\sigma^2}{2\alpha} e^{-\alpha(t-s)}$

$Y_t = \int_0^t X_s ds$ "Physical" Brownian motion

# Geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dB_t \qquad \mu = \text{drift} \quad \sigma = \text{volatility}$$

By Ito's formula

$$S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma B_t}$$

is the solution

$S_t \geq 0$ so it is (Samuelson) a better model of stock prices than $B_t$ (Bachelier)

Sometimes people write

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

but note that $\frac{dS_t}{S_t} \neq d \log S_t$

# Bessel process ($d = 2$)

Let $B_t = (B_t^1, B_t^2)$ be 2d Brownian motion starting at 0,

$$r_t = |B_t| = \sqrt{(B_t^1)^2 + (B_t^2)^2}.$$

By Ito's lemma,

$$dr_t = \frac{B^1}{|B|} dB^1 + \frac{B^2}{|B|} dB^2 + \frac{1}{2} \frac{1}{|B|} dt.$$

This is *not* a stochastic differential equation.

$$Y(t) = \int_0^t \frac{B^1}{|B|} dB^1 + \int_0^t \frac{B^2}{|B|} dB^2$$

Let $f(t, y)$ be a smooth function Use Itô's lemma. Intuitively

$$df(t, Y_t) = \partial_t f dt + \partial_y f dY + \frac{1}{2}\partial_y^2 f (dY)^2$$

$$\begin{aligned}
(dY)^2 &= (\frac{B^1}{|B|} dB^1 + \frac{B^2}{|B|} dB^2)^2 \\
&= \left(\frac{B^1}{|B|}\right)^2 (dB^1)^2 + 2\frac{B^1 B^2}{|B|^2} dB^1 dB^2 + \left(\frac{B^2}{|B|}\right)^2 (dB^2)^2 \\
&= dt
\end{aligned}$$

$$\begin{aligned}
f(t, Y_t) &= f(0, Y_0) + \int_0^t (\partial_t f + \frac{1}{2}\partial_y^2 f)(s, Y_s) ds \\
&\quad + \int_0^t \partial_y f \frac{B^1}{|B|} dB^1 + \int_0^t \partial_y f \frac{B^2}{|B|} dB^2
\end{aligned}$$

## Itô's lemma

$$dX_t = \sigma(t, X_t)dB_t + b(t, X_t)dt$$

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \left\{ \partial_s f(s, X_s) + \mathcal{L}f(s, X_s) \right\} ds \\ &+ \int_0^t \sum_{i,j=1}^d \sigma_{ij}(s, X_s) \frac{\partial}{\partial x_i} f(s, X_s) dB_s^j \end{aligned}$$

$$\mathcal{L}f(t, x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t, x) \frac{\partial^2 f}{\partial x_i \partial x_j}(t, x) + \sum_{i=1}^d b_i(t, x) \frac{\partial f}{\partial x_i}(t, x)$$

$$a_{ij} = \sum_{k=1}^d \sigma_{ik}\sigma_{jk} \qquad a = \sigma\sigma^T$$

$$dr_t = \frac{B^1}{|B|}dB^1 + \frac{B^2}{|B|}dB^2 + \frac{1}{2}\frac{1}{|B|}dt = dY_t + \frac{1}{2}r_t^{-1}dt$$

$$\begin{aligned} f(t, Y_t) &= f(0, Y_0) + \int_0^t (\partial_t f + \frac{1}{2}\partial_y^2 f)(s, Y_s)ds \\ &\quad + \int_0^t \partial_y f \frac{B^1}{|B|}dB^1 + \int_0^t \partial_y f \frac{B^2}{|B|}dB^2 \end{aligned}$$

In particular $e^{\lambda Y_t - \lambda^2 t/2}$ is a martingale

So $Y_t$ is a Brownian motion.

Therefore

$$dr_t = dY_t + \frac{1}{2}r_t^{-1}dt$$

*is* a stochastic differential equation for the new Brownian motion $Y_t$

## Itô's lemma

$$f(t, X_t) - f(0, X_0) = \int_0^t \left\{ \partial_s + \mathcal{L} \right\} f(s, X_s) ds + \int_0^t \nabla f(s, X_s) \cdot \sigma dB_s$$

## Proof

$$= \sum_i f(t_{i+1}, X_{t_{i+1}}) - f(t_i, X_{t_i})$$

$$= \sum_i \frac{\partial f}{\partial t}(t_i, X_{t_i})(t_{i+1} - t_i) + \nabla f(t_i, X_{t_i}) \cdot (X_{t_{i+1}} - X_{t_i})$$

$$+ \frac{1}{2} \sum_{j,k=1}^{d} \frac{\partial^2 f}{\partial x_j \partial x_k}(t_i, X_{t_i})(X^j_{t_{i+1}} - X^j_{t_i})(X^k_{t_{i+1}} - X^k_{t_i})$$

$$+ \text{ higher order terms}$$

## Proof continued

$$\sum_i \frac{\partial f}{\partial t}(t_i, X_{t_i})(t_{i+1} - t_i) \to \int_0^t \frac{\partial f}{\partial t}(s, X_s)ds$$

$$\sum_i \nabla f(t_i, X_{t_i}) \cdot (X_{t_{i+1}} - X_{t_i})$$

$$= \sum_i \nabla f(t_i, X_{t_i}) \cdot (\int_{t_i}^{t_{i+1}} \sigma(s, X_s)dB_s) \quad \to \int_0^t \nabla f \cdot \sigma dB$$

$$+ \sum_i \nabla f(t_i, X_{t_i}) \cdot (\int_{t_i}^{t_{i+1}} b(s, X_s)ds) \quad \to \int_0^t \nabla f \cdot b \, ds$$

$$\sum_i \frac{\partial^2 f}{\partial x_j \partial x_k}(t_i, X_{t_i})(X^j_{t_{i+1}} - X^j_{t_i})(X^k_{t_{i+1}} - X^k_{t_i}) \to \int_0^t \frac{\partial^2 f}{\partial x_j \partial x_k}(s, X_s)a_{jk}(s, X_s)ds$$

## Proof continued

To show the last convergence, ie

$$\sum_i g(t_i, X_{t_i})(X^j_{t_{i+1}} - X^j_{t_i})(X^k_{t_{i+1}} - X^k_{t_i}) \to \int_0^t g(s, X_s)a_{jk}(s, X_s)ds$$

$$\begin{aligned} Z(t_i, t_{i+1}) &= (\int_{t_i}^{t_{i+1}} \sum_l \sigma_{jl}(s, X_s)dB^l_s)(\int_{t_i}^{t_{i+1}} \sum_m \sigma_{km}(s, X_s)dB^m_s) \\ &\quad - \int_{t_i}^{t_{i+1}} \sum_l \sigma_{jl}\sigma_{kl}(s, X_s)ds \end{aligned}$$

$$E[|Z(t_i, t_{i+1})|^2] = \mathcal{O}((t_{i+1} - t_i)^2)$$

## Proof continued

$$\sum_i g(t_i, X_{t_i}) E[\int_{t_i}^{t_{i+1}} a_{ij}(s, X_s) ds] \to \int_0^t g(s, X_s) a_{ij}(s, X_s) ds$$

$$E[(\sum_i g(t_i, X_{t_i}) Z(t_i, t_{i+1}))^2] = \sum_{i,j} E[g(t_i, X_{t_i}) Z(t_i, t_{i+1}) g(t_j, X_{t_j}) Z(t_j, t_{j+1})]$$

$$i < j \quad E[E[g(t_i, X_{t_i}) Z(t_i, t_{i+1}) g(t_j, X_{t_j}) Z(t_j, t_{j+1}) \mid \mathcal{F}_{t_j}]] = 0$$

$$\begin{aligned} i = j \qquad & E[E[g^2(t_i, X_{t_i}) Z^2(t_i, t_{i+1}) \mid \mathcal{F}_{t_i}]] \\ & = E[g^2(t_i, X_{t_i}) E[Z^2(t_i, t_{i+1}) \mid \mathcal{F}_{t_i}]] \\ & = \mathcal{O}((t_{i+1} - t_i)^2) \end{aligned}$$

$$f(t,X_t)-f(0,X_0)-\int_0^t\Big\{\partial_s+\mathcal{L}\Big\}f(s,X_s)ds=\int_0^t\nabla f(s,X_s)\cdot\sigma dB_s$$

$$\mathcal{L}=\frac{1}{2}\sum_{i,j=1}^{d}a_{ij}(t,x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d}b_i(t,x)\frac{\partial}{\partial x_i}\quad = generator$$

$M_t=f(t,X_t)-\int_0^t\Big\{\partial_s+\mathcal{L}\Big\}f(s,X_s)ds$ is a martingale

$$\begin{aligned}0 &= E[f(t,X_t)-f(s,X_s)-\int_s^t\Big\{\partial_u+\mathcal{L}\Big\}f(u,X_u)du\mid\mathcal{F}_s]\\ &= \int f(t,y)p(s,x,t,y)dy-f(s,x)\\ &\quad -\int_s^t\int\Big\{\partial_u+\mathcal{L}\Big\}f(u,y)p(s,x,u,y)dydu,\qquad X_s=x\end{aligned}$$

For any $f$,

$$
\begin{aligned}
0 &= \int f(t,y)p(s,x,t,y)dy - f(s,x) \\
&\quad - \int_s^t \int \left\{\partial_u + \mathcal{L}\right\} f(u,y)p(s,x,u,y)dydu
\end{aligned}
$$

## Fokker-Planck (Forward) Equation

$$
\begin{aligned}
\frac{\partial}{\partial t} p(s,x,t,y) &= \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial y_i \partial y_j}(a_{i,j}(t,y)p(s,x,t,y)) \\
&\quad - \sum_{i=1}^{d} \frac{\partial}{\partial y_i}(b_i(t,y)p(s,x,t,y)) \\
&= L_y^* p(s,x,t,y)
\end{aligned}
$$

$$
\lim_{t \downarrow s} p(s,x,t,y) = \delta(y-x).
$$

## Kolmogorov (Backward) Equation

$$
\begin{aligned}
-\frac{\partial}{\partial s} p(s,x,t,y) &= \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(s,x)\frac{\partial^2 p(s,x,t,y)}{\partial x_i \partial x_j} \\
&\quad + \sum_{i=1}^{d} b_i(s,x)\frac{\partial p(s,x,t,y)}{\partial x_i} \\
&= L_x p(s,x,t,y)
\end{aligned}
$$

$$
\lim_{s\uparrow t} p(s,x,t,y) = \delta(y-x).
$$

## Proof.

$f(x)$ smooth

$$-\frac{\partial}{\partial s}u = L_s u \quad 0 \leq s < t \quad u(t,x) = f(x)$$

Ito's formula: $u(s, X(s))$ martingale up to time $t$

$$u(s,x) = E_{s,x}[u(s,X(s))] = E_{s,x}[u(t,X(t))] = \int f(z)p(s,x,t,z)dz$$

Let $f_n(z)$ smooth functions tending to $\delta(y-z)$. We get in the limit that $p$ satisfy the backward equations. □

## Example. Brownian motion $d = 1$

$$\mathcal{L} = \frac{1}{2}\frac{\partial^2}{\partial x^2}$$

$$\textit{Forward} \quad \frac{\partial p(s,x,t,y}{\partial t} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial y^2}, \quad t > s$$

$$p(s,x,s,y) = \delta(y-x)$$

$$\textit{Backward} \quad -\frac{\partial p(s,x,t,y}{\partial s} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial x^2}, \quad s < t,$$

$$p(t,x,t,y) = \delta(y-x)$$

## Example. Ornstein-Uhlenbeck Process

$$\mathcal{L} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - \alpha x \frac{\partial}{\partial x}$$

*Forward* $$\frac{\partial p(s,x,t,y}{\partial t} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial y^2} + \frac{\partial}{\partial y}(\alpha y p(s,x,t,y), \quad t > s,$$

$$p(s,x,s,y) = \delta(y-x)$$

*Backward* $$-\frac{\partial p(s,x,t,y}{\partial s} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial x^2} - \alpha x \frac{\partial p(s,x,t,y)}{\partial x}, \quad s < t,$$

$$p(t,x,t,y) = \delta(y-x)$$