Existence and Uniqueness Theorem

$\sigma : \mathbb{R}^d \times [0, T] \to \mathbb{R}^{d\times d}$, $b : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ be Borel measurable, $\exists A < \infty$,

$$\|\sigma(x,t)\| + |b(x,t)| \le A(1+|x|) \qquad x \in \mathbb{R}^d,\ 0 \le t \le T$$

and *Lipschitz*;

$$\|\sigma(x,t) - \sigma(y,t)\| + |b(x,t) - b(y,t)| \le A|x-y|.$$

$x_0 \in \mathbb{R}^d$ indep of B_t, $E[|x_0|^2] < \infty$.
Then there exists a unique solution X_t on $[0, T]$ to

$$dX_t = b(X_t,t)dt + \sigma(X_t,t)dB_t, \qquad X_0 = x_0$$

and $E[\int_0^T |X_t|^2 dt] < \infty$.

Uniqueness means that if X_t^1 and X_t^2 are two solutions then

$$P(X_t^1 = X_t^2,\ 0 \le t \le T) = 1$$

Proof of Uniqueness

Suppose X_t^1 and X_t^2 are solutions

$$
\begin{aligned}
X_t^1 - X_t^2 &= \int_0^t (b(X_s^1, s) - b(X_s^2, s))ds + \int_0^t (\sigma(X_s^1, s) - \sigma(X_s^2, s))dB_s \\
&\quad + x_0^1 - x_0^2
\end{aligned}
$$

$$
\begin{aligned}
E[|X_t^1 - X_t^2|^2] \leq 4E[|\int_0^t (b(X_s^1, s) - b(X_s^2, s))ds|^2] \\
+4E[|\int_0^t (\sigma(X_s^1, s) - \sigma(X_s^2, s))dB_s|^2] + 4E[|x_0^1 - x_0^2|^2]
\end{aligned}
$$

$$
E[|\int_0^t (b(X_s^1, s) - b(X_s^2, s))ds|^2] \leq A^2 \int_0^t E[|X_s^1 - X_s^2|^2]ds
$$

$$
\begin{aligned}
E[\int_0^t |\sigma(X_s^1, s) - \sigma(X_s^2, s))|^2 ds] &= E[|\int_0^t (\sigma(X_s^1, s) - \sigma(X_s^2, s))ds|^2] \\
&\leq A^2 \int_0^t E[|X_s^1 - X_s^2|^2]ds
\end{aligned}
$$

Call $\phi(t) = E[|X_t^1 - X_t^2|^2]$

$$\phi(t) \leq 8A^2 \int_0^t \phi(s)ds + 4\phi(0)$$

$$\Phi(t) = \int_0^t \phi(s)ds$$

$$(e^{-8A^2 t}\Phi(t))' = (\Phi'(t) - 8A^2\Phi(t))e^{-t} \leq 4\phi(0)e^{-t}$$

$$e^{-8A^2 t}\Phi(t) \leq 4\phi(0)$$

$$\phi(t) \leq 8A^2\Phi(t) \leq 4e^{8A^2 t}\phi(0)$$

$$E[|X_t^1 - X_t^2|^2] \leq 4e^{8A^2 t}E[|x_0^1 - x_0^2|^2]$$

For each $0 \leq t \leq T$, $X_t^1 = X_t^2$ a.s.so $X_t^1 = X_t^2$ for all rational $t \in [0, T]$ a.s. By continuity this implies that $X_t^1 = X_t^2$ for all $t \in [0, T]$ a.s.

Proof of Existence

$$X_0(t) \equiv x_0$$

$$X_n(t) = x_0 + \int_0^t \sigma(s, X_{n-1}(s))dB(s) + \int_0^t b(s, X_{n-1}(s))ds$$

$$\begin{aligned}
& E[\sup_{0\le t\le T} |X_n(t) - X_{n-1}(t)|^2] \\
\text{Doob's inequality} \quad & \le 4E[\int_0^T \|\sigma(s, X_{n-1}(s)) - \sigma(s, X_{n-2}(s))\|^2 ds] \\
& \quad + TE[\int_0^T |b(s, X_{n-1}(s)) - b(s, X_{n-2}(s))|^2 ds] \\
& \le C \int_0^T E[|X_{n-1}(s) - X_{n-2}(s)|^2] ds \\
& \le CE[\sup_{0\le t\le T} |X_{n-1}(t) - X_{n-2}(t)|^2]
\end{aligned}$$

Proof.

$$E[\sup_{0\le t\le T}|X_n(t)-X_{n-1}(t)|^2] \le CE[\sup_{0\le t\le T}|X_{n-1}(t)-X_{n-2}(t)|^2]$$

$$E[\sup_{0\le t\le T}|X_n(t)-X_{n-1}(t)|^2] \le \frac{(CT)^n}{n!}$$

$$P(\sup_{0\le t\le T}|X_n(t)-X_{n-1}(t)| > \frac{1}{2^n}) \le 2^{2n}E[\sup_{0\le t\le T}|X_n(t)-X_{n-1}(t)|^2]$$

summable

$$\text{Borel}-\text{Cantelli} \quad \Rightarrow \qquad P(\sup_{0\le t\le T}|X_n(t)-X_{n-1}(t)| > \frac{1}{2^n}\ i.o.) = 0.$$

Hence for almost every ω, $X_n(t) = X_0(t) + \sum_{j=0}^{n-1}(X_{j+1}(t) - X_j(t))$ converges uniformly on $[0, T]$ to a limit $X(t)$ which solves the required stochastic integral equation □

Lipschitz condition is *not* necessary

Theorem

Let $d = 1$ and

$$\begin{aligned} |b(t,x) - b(t,y)| &\leq C|x-y| \\ |\sigma(t,x) - \sigma(t,y)| &\leq C|x-y|^{\alpha}, \quad \alpha \geq 1/2 \end{aligned}$$

Then there exists a solution of $dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$ and it is unique

Example

$\sigma(x) = \mathrm{sgn}(x)$ and $dX = \sigma(B)dB$ Not a stochastic differential equation
But X is a Brownian motion $dB = \sigma(B)dX$ *is* a stochastic differential equation
But also $d(-B) = \sigma(-B)dX$ so no uniqueness

Markov property

X_t can be obtained by solving the stochastic differential equation up to time $s < t$ and then solving in $[s, t]$ with initial condition X_s
By uniqueness this gives the same answer
Define the transition probability

$$p(s, x, t, A) = P(X_t^{s,x} \in A)$$

where $X_t^{s,x}$ is the solution starting at x at time s
From the construction we have

$$P(X_t^{0,x} \in A \mid \mathcal{F}_s) = p(s, X_s^{0,x}, t, A)$$

which is the Markov property

Diffusions

A diffusion is a Markov process with transition probabilities $p(s, x, t, dy)$ satisfying, for each $\delta > 0$ as $h \to 0$,

$$i. \quad \frac{1}{h} \int_{|y-x| \geq \delta} p(t, x, t+h, dy) \to 0 \qquad \Rightarrow \text{ continuous paths}$$

$$ii. \quad \frac{1}{h} \int_{|y-x| < \delta} (y - x) p(t, x, t+h, dy) \to b(t, x)$$

$$iii. \quad \frac{1}{h} \int_{|y-x| < \delta} (y_i - x_i)(y_j - x_j) p(t, x, , t+h, dy) \to a_{ij}(t, x)$$

Formal derivation of the backward equation

$$p(s,x,t,A) = \int p(s,x,s+h,dy)p(s+h,y,t,A)$$

$$0 = \int p(s,x,s+h,dy)\Big\{p(s+h,y,t,A) - p(s,x,t,A)\Big\}$$

$$\begin{aligned} 0 &= \int p(s,x,s+h,dy)\Big\{h\frac{\partial p(s,x,t,A)}{\partial s} + \sum_{i=1}^{d}(y_i - x_i)\frac{\partial p(s,x,t,A)}{\partial x_i} \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{d}(y_i - x_i)(y_j - x_j)\frac{\partial^2 p(s,x,t,A)}{\partial x_i \partial x_j} + \cdots\Big\} \end{aligned}$$

$$-\frac{\partial p(s,x,t,A)}{\partial s} = \sum_{i=1}^{d} b_i(t,x)\frac{\partial p(s,x,t,A)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,x)\frac{\partial^2 p(s,x,t,A)}{\partial x_i \partial x_j}$$

Proof.

$f(x)$ smooth

$$-\frac{\partial}{\partial s}u = L_s u \quad 0 \leq s < t \quad u(t,x) = f(x)$$

Ito's formula: $u(s, X(s))$ martingale up to time t

$$u(s,x) = E_{s,x}[u(s,X(s))] = E_{s,x}[u(t,X(t))] = \int f(z)p(s,x,t,dz)$$

Let $f_n(z)$ smooth functions tending to $\delta(y-z)$

$$u(s,x) = p(s,x,t,y) \quad \text{if} \quad -\frac{\partial}{\partial s}u = L_s u \quad 0 \leq s < t \quad u(t,x) = \delta(x-y)$$

□

Existence result from PDE

Suppose that $a(t,x)$ and $b(t,x)$ are bounded and that there are $\alpha > 0$, $\gamma \in (0,1]$, $C < \infty$ such that for all $s,t \geq 0$, $x,y \in \mathbb{R}^d$,

i. $\quad \xi^T a(t,x)\xi \geq \alpha|\xi|^2, \quad \xi \in \mathbb{R}^d,$

ii. $\quad \|a(s,x) - a(t,y)\| + |b(s,x) - b(t,y)| \leq C(|x-y|^\gamma + |t-s|^\gamma).$

Then the backward equation has a solution and furthermore

$$p(s,x,t,A) = \int_A p(s,x,t,y)dy$$

with $p(s,x,t,y) \geq 0$ jointly continuous in s,x,t,y. Furthermore, $p(s,x,t,y)$ is the unique weak solution of the forward equation,i.e.

$$\int f(t,y)p(s,x,t,y)dy - f(s,x) = \int_s^t \int \{\partial_u + \mathcal{L}\} f(u,y)p(s,x,u,y)dydu$$

The solution X_t, $t \geq 0$ of $dX_t = b(X_t)dt + \sigma(X_t)dB_t$ with $X_0 = x$ is a Markov process with infinitesimal generator

$$L = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x)\frac{\partial}{\partial x_i}, \qquad a = \sigma\sigma^*.$$

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \Big\{\partial_s f(s, X_s) + \mathcal{L} f(s, X_s)\Big\} ds \\ &\quad + \int_0^t \sum_{i,j=1}^{d} \sigma_{ij}(s, X_s)\frac{\partial}{\partial x_i} f(s, X_s) dB_s^j \end{aligned}$$

Example. Brownian motion $d = 1$

$$\mathcal{L} = \frac{1}{2}\frac{\partial^2}{\partial x^2}$$

$$\textit{Forward} \quad \frac{\partial p(s,x,t,y}{\partial t} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial y^2}, \quad t > s$$

$$p(s,x,s,y) = \delta(y-x)$$

$$\textit{Backward} \quad -\frac{\partial p(s,x,t,y}{\partial s} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial x^2}, \quad s < t,$$

$$p(t,x,t,y) = \delta(y-x)$$

Example. Ornstein-Uhlenbeck Process

$$\mathcal{L} = \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} - \alpha x \frac{\partial}{\partial x}$$

$$\textit{Forward} \quad \frac{\partial p(s,x,t,y}{\partial t} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial y^2} + \frac{\partial}{\partial y}(\alpha y p(s,x,t,y), \quad t > s,$$

$$p(s,x,s,y) = \delta(y-x)$$

$$\textit{Backward} \quad -\frac{\partial p(s,x,t,y}{\partial s} = \frac{1}{2}\frac{\partial^2 p(s,x,t,y)}{\partial x^2} - \alpha x \frac{\partial p(s,x,t,y)}{\partial x}, \quad s < t,$$

$$p(t,x,t,y) = \delta(y-x)$$