PROBLEMS (due Apr 5)

1. A common model for interest rates is the *Vasicek model*, $dr(t) = (\theta - \alpha r(t))dt + \sigma dB(t)$.

   Relate it to the Ornstein-Uhlenbeck process.

   The *discount function* is

   $$Z_{t,T}(\omega) = E[e^{-\int_t^T r(s)ds} \mid \mathcal{F}(t)].$$

   i. Show that in fact $Z_{t,T}$ is only a function of $r(t)$ (which we may as well call $Z_{t,T}(r(t))$.) ii. Fix t and show that $Z_{t,T}(r)$ is the solution of the equation

   $$\frac{\partial Z}{\partial T} = (\theta - \alpha r)\frac{\partial Z}{\partial r} + \sigma^2 \frac{\partial^2 Z}{\partial r^2} - rZ$$

   with $Z_{t,t} = 1$. iii. Show that the continuously compounded interest rate $R_{t,T} = -(T-t)^{-1} \ln Z_{t,T}$ is of the special form $R(t,T) = a(T-t) + b(T-t)r(t)$ and find the functions $a(t)$ and $b(t)$. iv. Repeat i.-iii. for the CIR model $dr(t) = (\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}dB(t)$

2. Compute the mean $E[r(t)]$ and the variance $\mathrm{Var}(r(t))$ for the Vasicek and CIR models.

3. Let $X_1(\cdot)$ and $X_2(\cdot)$ solve the two constant coefficient sde's $dX_1(t) = bdt + \sigma_1 dB(t)$ and $dX_2(t) = bdt + \sigma_2 dB(t)$. How big is

   $$\frac{P(X_1(t_1) \in dx_1, \ldots, X_1(t_n) \in dx_n)}{P(X_2(t_1) \in dx_1, \ldots, X_2(t_n) \in dx_n)}$$

   as n becomes large?

4. If P and $\tilde{P}$ are equivalent and $\frac{d\tilde{P}}{dP} = Z$ show that $\frac{dP}{d\tilde{P}} = \frac{1}{Z}$.

   Let $P_x^{a,b}$ denote the probability measure on $C([0,T])$ corresponding to the solution of the stochastic differential equation

   $$dX(t) = \sigma(t, X(t))dB(t) + b(t, X(t))dt, \quad X(0) = x$$

   where $a = \sigma\sigma^T$. Let $b_1 \neq b_2$. Write expressions for $\frac{dP_x^{a,b_1}}{dP_x^{a,b_2}}$ and $\frac{dP_x^{a,b_2}}{dP_x^{a,b_1}}$ using the Cameron-Martin- Girsanov formula.

   Is the second the inverse of the first, or not? Find an explanation.

5. Let $\alpha(x) = (\alpha_1(x), ..., \alpha_n(x))$ be a smooth function from R^n to R^n . Consider the partial differential equation for $x \in R^n$, and $t > 0$,

   $$\frac{\partial u}{\partial t} = \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^{n} \alpha_i(x)\frac{\partial u}{\partial x_i}, \quad u(0,x) = f(x).$$

i. Use the Girsanov theorem to show that the solution is

$$u(t,x) = E_x[e^{\int_0^t \alpha(B(s))dB(s) ? \frac{1}{2}\int_0^t |\alpha(B(s))|^2 ds} f(B(t))].$$

ii. Suppose that $\alpha(x) = \nabla\gamma(x)$ for some function $\gamma : R^n \to R$. Use Ito's formula to show that in this case

$$u(t,x) = e^{-\gamma(x)} E_x[e^{\gamma(B(t)) ? \frac{1}{2}\int_0^t (\nabla\gamma^2(B(s)) + \Delta\gamma(B(s)))ds} f(B(t))].$$

iii. Use the Feynman-Kac formula to show that $v(t,x) = e^{\gamma(x)}u(t,x)$ is the solution of

$$\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v - \frac{1}{2}(\nabla\gamma^2 + \Delta\gamma)v.$$