LECTURE 10

Geometric Brownian motion. Let $B(t)$ be a Brownian motion. Then

$$S(t) = S(0)e^{\{\sigma B(t)+(\mu-\frac{1}{2}\sigma^2)t\}}$$

is called geometric Brownian motion. We assume that $S(0)$ is independent of $B(t)$. It is commonly used as a model for price processes. The mean and variance are easy to compute: Since we know that $e^{\{\alpha B(t)-\frac{1}{2}\alpha^2 t\}}$ is a martingale for any α we have

$$E[S(t)] = E[S(0)e^{\mu t}E[e^{\{\sigma B(t)-\frac{1}{2}\sigma^2)t\}} \mid \mathcal{F}(0)]] = e^{\mu t}E[S(0)]$$

and

$$E[S^2(t)] = E[S^2(0)e^{(2\mu+\frac{3}{2}\sigma^2)t}E[e^{\{2\sigma B(t)-2\sigma^2)t\}} \mid \mathcal{F}(0)]] = e^{(2\mu+\frac{3}{2}\sigma^2)t}E[S^2(0)].$$

We apply Ito's formula to the function $f(x,t) = e^{\{\sigma x+(\mu-\frac{1}{2}\sigma^2)t\}}$. Since f satisfies the equations

$$\frac{\partial f}{\partial x} = \sigma f, \qquad \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = \mu f$$

we have

$$S(t) = S(0) + \int_0^t \mu S(u)du + \int_0^t \sigma S(u)dB(u)$$

which we can write in differential notation as

$$dS = \mu S dt + \sigma S dB.$$

So we solved a stochastic differential equation. Geometric Brownian motion is the solution of $dS = \mu S dt + \sigma S dB$.

Stochastic differential equations. A stochastic differential equation is an expression of the form

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t), \qquad X(0) = x. \qquad (*)$$

In one dimension b and σ are scalars. In n dimensions, b should be an n-vector, σ should be an $n \times m$ matrix and $B(t)$ should be an m dimensional Brownian motion. In terms of notation, everybody seems to agree on σ, but drift term, which I have called b, is called all sorts of things in the finance literature. All probabilists call it b, so I will stick with that.

Diffusions. The solution of the stochastic differential equation will be a diffusion process starting at x which is a continuous strong Markov process which looks 'locally' like a Brownian motion with covariance $a = \sigma\sigma^T$ and drift b. More precisely, for any twice continuously differentiable function f,

$$\lim_{t\downarrow 0} \frac{E_x[f(X(t))] - f(x)}{t} = Lf(x)$$

where

$$Lf(x) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(t,x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} b_i(t,x)\frac{\partial f}{\partial x_i}(x). \tag{$\spadesuit$}$$

Euler approximations. If $Y_1, Y_2, \ldots$ are independent standard normal random variables (mean 0 and variance 1), then if you took some step size $1/n$ and made the Euler approximation

$$X^n(\frac{i+1}{n}) = X^n(\frac{i}{n}) + b(\frac{i}{n}, X^n(\frac{i}{n}))\frac{1}{n} + \sigma(\frac{i}{n}, X^n(\frac{i}{n}))\frac{Y_i}{\sqrt{n}}$$

then X^n will be a good approximation of X.

Stochastic integral equation. The equation $(*)$ by itself does not mean very much $(0=0)$. Unlike with ordinary differential equations we do NOT have the luxury of rewriting it as $\frac{dX(t)}{dt} = b(t, X(t)) + \sigma(t, X(t))\frac{dB(t)}{dt}$. This is really nonsense because $B(t)$ and $X(t)$ are non-differentiable at every point with probability one. What $(*)$ means is a shorthand for

$$X(t) = x + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dB(s).$$

The first is a normal Riemann integral and the second is a stochastic integral. The primary objects in stochastic calculus are stochastic integrals and the differentials are only defined in terms of them.

Ito's formula for solutions of stochastic differential equations. Suppose that $X(t)$ is the solution of $dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t)$. Then a computation identical to the derivation of the Ito formula for Brownian motion gives

$$f(X(t), t) - f(X(0), 0) = \int_0^t \sigma\nabla f \cdot dB(s) + \int_0^t [\frac{\partial f}{\partial t} + Lf]ds$$

where L is given in $(\spadesuit)$.

Feynman-Kac formula. Let $u(t,x)$ be the solution of the equation

$$\frac{\partial u}{\partial t} = Lu + V(x)u, \qquad u(0,x) = f(x).$$

Then u has the representation

$$u(T,x) = E_x[f(X(T))e^{\int_0^T V(X(s))ds}]$$

where E_x denotes expectation with respect to the solution of $dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t)$ with $X(0) = x$.

Proof. Apply Ito's formula to see that $u(T-t, X(t))e^{\int_0^t V(X(s)ds}$ is a martingale. Take expectations at $t=0$ and $t=T$ to get the Feynman-Kac formula.

Application to Cox-Ingersol-Ross model. The interest rate $r(t)$ is assumed to satisfy the equation

$$dr(t) = (\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}dB(t).$$

Let's calculate the mean. Since the last term is a martingale we have

$$E[r(t)] = E[r(0)] + \int_0^t (\alpha - \beta E[r(s)])ds.$$

Let $m(t) = E[r(t)]$. Differentiating gives

$$m' = \alpha - \beta m$$

which has the unique solution

$$E[r(t)] = m(t) = e^{-\beta t} m(0) + (1 - e^{-\beta t})\alpha/\beta.$$

To compute the variance we use Ito's formula. Let $f(x) = x^2$. The operator $Lf = \frac{1}{2}\sigma^2 x f''(x) + (\alpha - \beta x)f'(x) = \sigma^2 x + (\alpha - \beta x)(2x)$ so we have

$$r^2(t) = r^2(0) + 2\sigma \int_0^t r^{3/2}(s)dB(s) + \int_0^t (-2\beta r^2(s) + [\sigma^2 + \alpha]r(s))ds.$$

Taking expectations and letting $f(t) = E[r^2(t)]$ we get

$$f(t) = f(0) - 2\beta \int_0^t f(s)ds + [\sigma^2 + \alpha] \int_0^t m(s)ds.$$

Since we know $m(t)$ already we can plug it in and differentiate to get a differential equation for $f(t)$,

$$f' = 2\beta f + [\sigma^2 + \alpha][e^{-\beta t} m(0) + (1 - e^{-\beta t})\alpha/\beta]$$

which is straightforward to solve.

Existence and uniqueness theorem for stochastic differential equations. One of first things one wants to know about the stochastic differential equation $(*)$ is whether there really exists a solution. It is only in very rare circumstances that you have an explicit solution, so you need a theorem which says under some reasonable conditions there is one, even if you don't have an exact formula for it. Then maybe you can still calculate whatever quantity interests you even without explicit formulas. On the other hand even in the lucky case where we can write down a solution, we still need to know that there are no other pretenders lurking out there. That is the business of the uniqueness theorem. The classical result is

Theorem. *Suppose that b and σ are Lipschitz functions in space, i.e. there exists some constant C so that*

$$|b_i(t,x) - b_i(t,y)| \le C|x-y|, \qquad |\sigma_{ij}(t,x) - \sigma_{ij}(t,y)| \le C|x-y|.$$

Then there exists a solution of the stochastic differential equation $(*)$, and that solution is unique.

Proof. The proof is by Picard iteration and it is almost exactly the same as the proof of the corresponding existence and uniqueness theorem for ordinary differential equations (which, after all, is just the special case $\sigma = 0$).

Existence. Consider the equation on a finite time interval $[0, T]$. Define $X_0(t) = x$ and by induction

$$X_n(t) = x + \int_0^t \sigma(s, X_{n-1}(s))dB(s) + \int_0^t b(s, X_{n-1}(s))ds.$$

These are the Picard iterates. You can see they are all perfectly well defined continuous functions, and if we can show they have some decent limit, then that limit should satisfy the stochastic integral equation.

For any martingale $M(t)$ there is a very useful inequality called *Doob's inequality* which says that

$$E[\sup_{0\le t\le T} |M(t)|^2] \le 4E[|M(T)|^2].$$

The stochastic integral in the definition of the Picard iterates is of course a martingale so we can apply Doob's inequality, and the L^2 isometry, and after a little futzing around we get

$$\begin{aligned} E[\sup_{0\le t\le T} |X_n(t) - X_{n-1}(t)|^2] \le & 4E[\int_0^T \|\sigma(s, X_{n-1}(s)) - \sigma(s, X_{n-2}(s))\|^2 ds] \\ & + TE[\int_0^T |b(s, X_{n-1}(s)) - b(s, X_{n-2}(s))|^2 ds]. \end{aligned}$$

Now we can use the Lipschitz bounds to get, for some $C < \infty$,

$$E[\sup_{0\le t\le T} |X_n(t) - X_{n-1}(t)|^2] \le CE[\sup_{0\le t\le T} |X_{n-1}(t) - X_{n-2}(t)|^2].$$

By induction this gives

$$E[\sup_{0\le t\le T} |X_n(t) - X_{n-1}(t)|^2] \le (CT)^n/n!.$$

Therefore

$$E[\sup_{0\le t\le T} |X_n(t) - X_m(t)|^2] \to 0 \qquad \text{as } n, m \to \infty.$$

In other words we proved the sequence is Cauchy. Therefore there exists a limit $X(t)$ which is clearly a solution of the stochastic integral equation on $[0, T]$. As $T \to \infty$ the solutions we construct are consistent with each other, so this gives the solution on $[0, \infty)$.

Uniqueness. If $X(t)$ and $\tilde{X}(t)$ are two solutions, then subtracting the two integral equations, applying the L^2 isometry, and using the Lipschitz property we get

$$E[|X(t) - \tilde{X}(t)|^2] \le CE[\int_0^t |X(s) - \tilde{X}(s)|^2 ds].$$

By induction,

$$E[|X(t) - \tilde{X}(t)|^2] \leq (Ct)^n/n!.$$

Letting $n \to \infty$ we get $X(t) = \tilde{X}(t)$.

Example: Ornstein-Uhlenbeck process. This is the solution of the *Langevin* equation

$$dX(t) = -\alpha X(t)dt + \sigma dB(t), \qquad X(0) = x$$

where α and σ are constants. This corresponds to $b(t,x) = -\alpha x$ and $\sigma(t,x) = \sigma$ which are both obviously Lipschitz functions, so there exists a solution and it is unique by the theorem. This is one of the few cases where the solution can be written down explicitly. It is easy to check that

$$X(t) = xe^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} dB(s)$$

works. This is a Gaussian process, so everything is determined by the mean $E[X(t)] = E[X(0)]e^{-\alpha t}$ and the covariance $\mathrm{Cov}(X(s), X(t)) = e^{-\alpha(t+s)}\{\mathrm{Cov}(X(0), X(0)) + \frac{\sigma^2}{2\alpha}(e^{2\alpha(t\wedge s)} - 1)\}$.

The $-\alpha X(t)dt$ term in the Langevin equation is a mean restoring force, proportional to the displacement. The mean here is just 0. Plus you force the thing with a white noise that doesn't depend on the solution. The equilibrium state is a normal distribution with mean zero and variance $\sigma^2/2\alpha$.

- **Vasicek model.** The short rate $r(t)$ is assumed to follow the sde

  $$dr(t) = (\theta - \alpha r(t))dt + \sigma dB(t).$$

  Clearly

  $$X(t) = r(t) - \theta/\alpha$$

  is the Ornstein-Uhlenbeck process. So you can just look up the solution above.

Strong and weak solutions. The type of solutions dealt with in the previous existence/uniqueness theorem are strong solutions. This means that you are given the Brownian motion $B(t)$ and asked to come up with a solution $X(t)$ of the stochastic differential equation $dX = bdt + \sigma dB$. A weak solution is when you just find some Brownian motion $\tilde{B}(t)$ for which you can solve the equation $dX = bdt + \sigma d\tilde{B}$. The difference may appear a bit subtle so we illustrate it with an example.

Example: Cox-Ingersol-Ross model. Vasicek/Hull-White has some problems as an interest rate model because $r(t)$ can be negative. The CIR model is the simplest model which avoids this. The short rate $r(t)$ is assumed to follow the sde

$$dr(t) = (\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}dB(t).$$

Note that the Lipschitz condition is not satisfied and the existence/uniqueness theorem does not apply. If $d = 4\alpha/\sigma^2$ is a positive integer, then we can find a solution as follows. Let

$B_1(t), B_2(t), \ldots, B_d(t)$ be d independent Brownian motions and let $X_1(t), X_2(t), \ldots, X_d(t)$ be the solutions of the Langevin equations

$$dX_i = -\alpha X_i dt + \sigma dB_i, \qquad i = 1, \ldots, d.$$

Let

$$r(t) = X_1^2(t) + \cdots + X_d^2(t).$$

By Ito's formula,

$$dr(t) = (\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}\left\{\sum_{i=1}^{d} \frac{X_i(t)}{\sqrt{r(t)}} dB_i(t)\right\}.$$

The last term

$$d\tilde{B}(t) = \sum_{i=1}^{d} \frac{X_i(t)}{\sqrt{r(t)}} dB_i(t)$$

is a martingale with quadratic variation

$$d\tilde{B}d\tilde{B} = \sum_{i,j=1}^{d} \frac{X_i(t)X_j(t)}{r(t)} dB_i(t)dB_j(t) = \sum_{i=1}^{d} \frac{X_i^2(t)}{r(t)} dt = dt$$

and hence $\tilde{B}(t)$ is a Brownian motion and we have

$$dr(t) = (\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}d\tilde{B}(t).$$

If $d = 1$ then $\tilde{B}(t) = B_1(t)$ so we have a strong solution. But if $d = 2, 3, \ldots$ then all we have is a weak solution, because given a Brownian motion $B(t)$ it is not at all clear how to find $B_1(t), \ldots, B_d(t)$ for which $dB(t) = \sum_{i=1}^{d} \frac{X_i(t)}{\sqrt{r(t)}} dB_i(t)$.

Levy's theorem. Actually if you look in the argument above we really used the following theorem when we identified $\tilde{B}(t)$ as a Brownian motion:

Theorem. *Let $B(t)$ be a process adapted to a filtration $\mathcal{F}(t)$ which*

i. has continuous sample paths

ii. is a martingale

iii. has quadratic variation t. Then $B(t)$ is a Brownian motion.

Proof. It is enough to show that for any α

$$e^{\alpha B(t) - \frac{1}{2}\alpha^2 t}$$

is a martingale because for any $t > s \geq 0$, using the martingale property and rearranging things a bit

$$E[e^{\alpha(B(t)-B(s))} \mid \mathcal{F}(s)] = e^{\frac{1}{2}\alpha^2(t-s)}$$

which says that 1. $B(t) - B(s)$ is independent of $\mathcal{F}(s)$ (since the right hand side is a constant); and 2. $B(t) - B(s)$ is $\mathcal{N}(0, t-s)$. That is enough to pin down $B(t)$ as Brownian motion. To prove it is a martingale, let $f(x,t) = e^{\alpha x - \frac{1}{2}\alpha^2 t}$. Then $\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = 0$ and hence by Ito's formula $f(B(t),t)$ is a martingale. That does it.

Problems.

1. Let $B_1(t)$ and $B_2(t)$ be two independent Brownian motions.
   i. Show that the Bessel process $X(t) = \sqrt{(B_1(t))^2 + (B_2(t))^2}$ satisfies a stochastic differential equation of the form

   $$dX(t) = \frac{1}{2X(t)}dt + dB(t)$$

   for some new Brownian motion $B(t)$.

   ii. In what qualitative way does the behaviour of the Bessel process differ from that of the CIR model? Why do you think the CIR model is a better model for interest rates? How will the behaviour of the CIR model differ from that of the Vasicek model? Why is it a better model for interest rates?
2. Consider the Vasicek model $dr(t) = (\theta - \alpha r(t))dt + \sigma dB(t)$. The *discount function* is

   $$Z_{t,T}(\omega) = E[e^{-\int_t^T r(s)ds} \mid \mathcal{F}(t)].$$

   i. Show that in fact $Z_{t,T}$ is only a function of $r(t)$ (which we may as well call $Z_{t,T}(r(t))$.)

   ii. Fix t and show that $Z_{t,T}(r)$ is the solution of the equation

   $$\frac{\partial Z}{\partial T} = (\theta - \alpha r)\frac{\partial Z}{\partial r} + \sigma^2 \frac{\partial^2 Z}{\partial r^2} - rZ$$

   with $Z_{t,t} = 1$.

   iii. Show that the continuously compounded interest rate $R_{t,T} = -(T-t)^{-1}\ln Z_{t,T}$ is of the special form $R(t,T) = a(T-t) + b(T-t)r(t)$ and find the functions $a(t)$ and $b(t)$. (cf. 17.4 of Hull, but if you use his formulae you have to verify that they really work.)
3. Repeat i.-iii. of 1. for the CIR model $dr(t) = (\alpha - \beta r(t))dt + \sigma\sqrt{r(t)}dB(t)$.