LECTURE 13

Brownian motion as limit of random walks. Let $X_1, X_2, \ldots$ be independent random variables with $P(X_i = 1) = P(X_i = -1) = 1/2$. Let $M_n = X_1 + \cdots + X_n$ and

$$B^{(n)}(t) = \frac{M_{\lfloor tn \rfloor}}{\sqrt{n}}.$$

In other words the random walk is taking steps of size $1/\sqrt{n}$ at times $1/n, 2/n, \ldots$ Then $B^{(n)}(t)$ converges to Brownian motion $B(t)$ as $n \to \infty$. To prove this: By the multidimensional central limit theorem it is clear that $(B^{(n)}(t_1), \ldots, B^{(n)}(t_k))$ is converging to a k dimensional Gaussian variable as $n \to \infty$. It only remains to check the covariance. If $t > s$ then

$$E[B^{(n)}(t)B^{(n)}(s)] = \frac{\lfloor ns \rfloor}{n}$$

which clearly converges to s as n gets large.

Brownian motion with variance σ^2 and drift μ as the limit of random walks. Let $X^{(n)}(t)$ be a random walk which at each time increment $1/n$ jumps $\frac{\sigma}{\sqrt{n}} + \frac{\mu}{n}$ or $-\frac{\sigma}{\sqrt{n}} + \frac{\mu}{n}$, each with probability $1/2$. Note that $X^{(n)}(t) - \frac{\mu}{n}\lfloor nt \rfloor$ is equal to $\sigma B^{(n)}(t)$ where $B^{(n)}(t)$ is as above. Therefore $X^{(n)}(t)$ will converge to $\sigma B(t) + \mu t$.

Geometric Brownian motion as limit of the binomial model. For each integer n consider a binomial model $S^{(n)}(t)$ with $u_n = 1 + \frac{\sigma}{\sqrt{n}}$ and $d_n = 1 - \frac{\sigma}{\sqrt{n}}$ and $r_n = 0$ and step size $1/n$. The risk neutral probabilities are q and $1 - q$ where $q = \frac{1+r_n-d_n}{u_n-d_n} = \frac{1}{2}$.

We want to show that $S^{(n)}(t) \to e^{\sigma B(t) - \frac{1}{2}\sigma^2 t}$ as $n \to \infty$.

Now at each time increment $1/n$, $\log S^{(n)}(t)$ jumps up $\log u_n$ or down $\log d_n$, each with probability $1/2$. Since $\log(1+x) = x - \frac{1}{2}x^2 + o(x^2)$ we have $\log u_n = \log(1 + \frac{\sigma}{\sqrt{n}}) = \frac{\sigma}{\sqrt{n}} + \frac{\sigma^2}{2n} + o(n)$ and $\log d_n = \log(1 - \frac{\sigma}{\sqrt{n}}) = -\frac{\sigma}{\sqrt{n}} + \frac{\sigma^2}{2n} + o(n)$.

By the previous subsection we have that $\log S^{(n)}(t) \to \sigma B(t) - \frac{1}{2}\sigma^2 t$, as desired.

Convergence of Markov chain approximations via martingales. The above derivations assumed that we already had an explicit formula for the desired diffusion in terms of Brownian motion. If we wanted a Markov chain approximation to a more general diffusion this type of analysis would not go very far.

Here is another proof that $B^{(n)}(t) = \frac{M_{\lfloor tn \rfloor}}{\sqrt{n}}$. converges to Brownian motion, which does extend to the general case. Recall that $B(t)$ is Brownian motion if and only if for every smooth bounded function f,

$$f(B(t)) - \int_0^t \frac{1}{2} f''(B(s))ds$$

is a martingale. For if this is true then $E[f(B(t)) - f(B(0) - \int_0^t \frac{1}{2}f''(B(s))ds] = 0$ or $\int f(y)p(0,0,t,y)dy - f(0) - \int_0^t \int \frac{1}{2}f''(z)p(0,0,s,z)dzds = 0$ which gives the forward equation $\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}$ after an integration by parts. So the fact that $f(B(t)) - \int_0^t \frac{1}{2}f''(B(s))ds$ is a martingale for every nice f is just another way of writing the forward equation.

Now (cf. lecture 4),

$$f(B^{(n)}(t)) - \frac{1}{n} \sum_{i=0}^{\lfloor nt \rfloor -1} L^{(n)} f(B^{(n)}(\frac{i}{n}))$$

is a martingale, where

$$L^{(n)} f(x) = n \left(\frac{1}{2} f(x + \frac{1}{\sqrt{n}}) + \frac{1}{2} f(x - \frac{1}{\sqrt{n}}) - f(x) \right).$$

An easy Taylor's expansion shows that for any nice function

$$L^{(n)} f(x) \to \frac{1}{2} f''(x) \qquad \text{as} \;\; n \to \infty$$

and hence

$$\frac{1}{n} \sum_{i=0}^{\lfloor nt \rfloor -1} L^{(n)} f(B^{(n)}(\frac{i}{n})) \to \int_0^t \frac{1}{2} f''(B(s)) ds.$$

In other words we have shown that the limiting process satisfies the correct forward equation.

This method is much better because it doesn't rely on explicit solutions.

Let's go through it again for the Binomial model's convergence to geometric Brownian motion.

Let $S^{(n)}(t)$ be the binomial model described above. We have

$$f(S^{(n)}(t)) - \frac{1}{n} \sum_{i=0}^{\lfloor nt \rfloor -1} L^{(n)} f(S^{(n)}(\frac{i}{n}))$$

is a martingale, where

$$L^{(n)} f(x) = n \left(\frac{1}{2} f(u_n x) + \frac{1}{2} f(d_n x) - f(x) \right) = n \left(\frac{1}{2} f((1 + \frac{\sigma}{\sqrt{n}})x) + \frac{1}{2} f((1 + \frac{\sigma}{\sqrt{n}})x) - f(x) \right).$$

A Taylor expansion gives

$$L^{(n)} f(x) \to \sigma^2 x^2 f''(x) = Lf(x) \qquad \text{as} n \to \infty$$

and hence in the limit we have

$$f(S(t)) - \int_0^t Lf(S(u)) du$$

is a martingale for every nice f, or in other words the transition probabilities satisfy the forward equation $\frac{\partial p}{\partial t} = L^* p$, identifying the process as the solution of the stochastic differential equation

$$dS(t) = \sigma S(t) dB(t),$$

or, in other words, $S(t)$ is geometric Brownian motion.

Problems.

1. Let V be the value of a European option where the interest rate is r and the underlying stock price process satisfies $dS = \mu S dt + \sigma S dB$. Let $\Gamma = \frac{\partial^2 V}{\partial S^2}$ and *vega* $v = \frac{\partial V}{\partial \sigma}$. Show that v satisfies the partial differential equation

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + rS\frac{\partial v}{\partial S} - rv + \sigma S^2 \Gamma = 0.$$

   Find the correct boundary condition at the expiry date.

2. Show that if the stock price process follows $dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dB(t)$, the interest rate is $r(t)$, and the stock pays continuously dividends at rate $D(t)$ where $\mu(t), \sigma(t), r(t)$ and $D(t)$ are known (non-random) functions of time, then the value at time 0 of a European call option with strike price K at time T is

$$\begin{aligned} &Se^{-\int_0^T D(u)du}\Phi\left(\frac{\log\left(\frac{S}{K}\right) + \int_0^T (r(u) - D(u))du + \frac{1}{2}\int_0^T \sigma^2(u)du}{\sqrt{\int_0^T \sigma^2(u)du}}\right) \\ &- Ke^{-\int_0^T r(u)du}\Phi\left(\frac{\log\left(\frac{S}{K}\right) + \int_0^T (r(u) - D(u))du - \frac{1}{2}\int_0^T \sigma^2(u)du}{\sqrt{\int_0^T \sigma^2(u)du}}\right). \end{aligned}$$

3.
   i. Construct a random walk model which converges to the solution of the stochastic differential equation $dX(t) = \sigma(t)dB(t) + \mu(t)dt$ where $\sigma(t)$ and $\mu(t)$ are given smooth non-random functions.
   ii. Construct a random walk model which converges to the solution of the stochastic differential equation $dS(t) = \sigma(t)S(t)dB(t) + \mu(t)S(t)dt$ where $\sigma(t)$ and $\mu(t)$ are given smooth non-random functions.

4. Consider the Ehrenfests' model: An urn contains n balls. m of the balls are white, and the rest are black. At each time step a ball is chosen at random from the urn and replaced with one of the opposite colour. Let X_k^n be the number of white balls minus the number of black balls after k steps.
   i. Write down the transition probabilities for the Markov chain.
   ii. Show that $Y^{(n)}(t) = \frac{X^n_{\lfloor nt \rfloor}}{\sqrt{n}}$ converges to the Ornstein-Uhlenbeck process.