LECTURE 14

Behaviour of stochastic differential equations under time change. Suppose that $X(t)$ is the solution of a one-dimensional stochastic differential equation $dX(t) = \sigma(t, X(t))dB(t) + b(t, X(t))dt$. Ito's formula tells us that $Y(t) = f(X(t))$ also satisfies a stochastic differential equation $dY(t) = \widetilde{\sigma}(t, Y(t))dB(t) + \widetilde{b}(t, Y(t))dt$ with appropriate $\widetilde{\sigma}$ and $\widetilde{b}$.

Now we ask about $Z(\tau) = X(t(\tau))$ where $t(\tau)$ is some given time change, i.e. $t(\tau)$ is a increasing function mapping $[0, \infty)$ into itself. For small h we have

$$
\begin{aligned}
Z(\tau + h) - Z(\tau) &= X(t(\tau + h)) - X(t(\tau)) \\
&\simeq \mathcal{N}((t(\tau + h) - t(\tau))b(t(\tau), X(t(\tau))), (t(\tau + h) - t(\tau))\sigma^2(t(\tau), X(t(\tau))) \\
&\simeq \mathcal{N}(h\widetilde{b}(\tau, Z(\tau)), h\widetilde{\sigma}^2(\tau, Z(\tau)))
\end{aligned}
$$

where

$$
\widetilde{b}(\tau, Z) = b(t(\tau), Z)/t'(\tau), \qquad \sigma(\tau, Z) = \sigma(t(\tau), Z)/\sqrt{t'(\tau)}.
$$

Hence $Z(\tau)$ is the solution of the stochastic differential equation

$$
dZ(\tau) = \widetilde{\sigma}(\tau, Z(\tau))d\widetilde{B} + \widetilde{b}(\tau, Z(\tau))d\tau
$$

where $\widetilde{B}$ is a Brownian motion.

This is valid also for *random* time changes $t(\tau)$ defined by $t(\tau) = \int_0^\tau f(u)du$ where f is adapted to the original Brownian filtration.

Application of time change to solve one dimensional stochastic differential equations. Suppose you want to solve the one dimensional stochastic differential equation $dX(t) = \sigma(t, X(t))dB(t) + b(t, X(t))dt$. By Cameron-Martin-Girsanov, you could instead solve $dZ(t) = \sigma(t, Z(t))dB(t)$ and then change to an equivalent measure which will correspond to the solution of $X(t)$. So it is enough to solve for $Z(t)$. What do we mean by *solve*? - the existence and uniqueness theorem tells us that (under reasonable conditions) there is a solution. What we want here is an explicit representation. If we define $t(\tau)$ by $\tau(t) = \int_0^t \sigma^2(u, Y(u))du$ then the above discussion shows that then $Y(\tau) = Z(t(\tau))$ is a Brownian motion and $Z(t)$ can be written explicitly as $Y(\tau(t))$.

This method doesn't really extend to multidimensional diffusions except in the special case where the diffusion matrix a is diagonal.

Application of time change to calibration of CIR model. We consider time dependent CIR models

$$
dr(t) = (\alpha(t) - \beta(t)r(t))dt + \sigma(t)\sqrt{r(t)}dB(t).
$$

The price of a zero coupon bond is

$$
Z(r(t), t, T) = E[e^{-\int_t^T r(u)du} \mid \mathcal{F}(t)].
$$

By the Feynman-Kac formula $Z(r,t,T)$ satisfies the partial differential equation

$$\frac{\partial Z}{\partial t} + (\alpha(t) - \beta(t) r)\frac{\partial Z}{\partial r} + \frac{1}{2}\sigma^2(t) r \frac{\partial^2 Z}{\partial r^2} - rZ = 0$$

on $[0,T)$ with final condition $Z(r,T,T) = 1$.

There is an explicit solution in the special case where α, β, and σ are constants given by

$$\begin{aligned} Z(r,t,T) &= e^{-rC(T-t) - A(T-t)} \\ C(t) &= \frac{\sinh(\gamma t)}{\gamma \cosh(\gamma t) + \frac{1}{2}\beta \sinh(\gamma t)} \\ A(t) &= -\frac{2\alpha}{\sigma^2} \log\left(\frac{\gamma e^{\beta t/2}}{\gamma \cosh(\gamma t) + \frac{1}{2}\beta \sinh(\gamma t)} \right) \end{aligned}$$

where $\gamma = \frac{1}{2}\sqrt{\beta^2 + 2\sigma^2}$.

We consider the calibration problem. $Z(r,0,T)$ is observed from market data and we want to find the best $\alpha(t)$, $\beta(t)$ and $\sigma(t)$ based on that observation. Since an explicit solution for Z in terms of $\alpha(t)$, $\beta(t)$, and $\sigma(t)$ does not exist except when they are time independent, this is a very difficult problem. To get around it people first find the computed Z as above for time independent coefficients, which we will call $Z^{CIR}(r,0,T)$ which is closest to the observed $Z(r,0,T)$. In other words we minimize over α, β and σ the distance between $Z^{CIR}(r,0,T) = e^{-rC(T)-A(T)}$ and the observed $Z(r,0,T)$ where A and C are as above and the distance is chosen as, say $\sup_{T\in[0,T_0]} |Z^{CIR}(1,0,T) - Z(1,0,T)|$ or whatever seems appropriate. Now the bond prices are all monotonic increasing functions of T, and therefore there exists a change of time variable $\tau(T)$ so that $Z^{CIR}(1,0,\tau(T)) = Z(1,0,T)$. The corresponding CIR model is $\hat{r}(\tau) = r(t(\tau))$ where $dr(t) = (\alpha^* - \beta^* r(t))dt + \sigma^* \sqrt{r(t)} dB(t)$ and α^*, β^* and σ^* are the minimizers above. The time changed process then satisfies

$$d\hat{r}(\tau) = (\hat{\alpha}(\tau) - \hat{\beta}(\tau)\hat{r}(\tau))d\tau + \hat{\sigma}(\tau)\sqrt{\hat{r}(\tau)} d\hat{B}(\tau)$$

where $\hat{\alpha}(\tau) = \alpha^*/t'(\tau)$, $\hat{\beta}(\tau) = \beta^*/t'(\tau)$ and $\hat{\sigma}(\tau) = \sigma^*/\sqrt{t'(\tau)}$.

Poisson process. Let $X_1, X_2, \ldots$ be independent identically distributed (i.i.d) random variables with exponential distribution

$$P(X_i \geq x) = e^{-\lambda x}.$$

In other words the probability density is $\lambda e^{-\lambda x}\mathbf{1}(x \geq 0)$. The expectation of each X_i is easily calculated to be λ^{-1} and the variance is λ^{-2}. The parameter $\lambda > 0$ is called the *intensity*. For $n = 1, 2, \ldots$, let

$$S_n = X_1 + \cdots + X_n.$$

Define $N(t)$ to be the number of indices k such that $S_k \leq t$. $N(t)$ is called the Poisson process. In other words, $N(t)$ is a continuous time process, taking only integer values. It waits at n for an exponential amount of time X_{n+1}, then jumps to $n+1$.

One dimensional marginals of the Poisson process. $N(t)$ has the Poisson distribution

$$P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

Proof. Clearly $P(N(t) = n) = P(S_n \leq t, S_{n+1} > t) = G_n(t) - G_{n+1}(t)$ where $G_n(t) = P(S_n \leq t)$. So it is equivalent to showing that

$$G_n(t) = 1 - e^{-\lambda t}\left(1 + \frac{\lambda t}{1!} + \cdots + \frac{(\lambda t)^{n-1}}{(n-1)!}\right)$$

which is equivalent to showing that the density $g_n(t) = G_n'(t)$ is given by

$$g_n(t) = \lambda \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t}. \tag{$*$}$$

For $n = 1$ this is just the definition of the exponential distribution. The density g_{n+1} is defined by the convolution

$$g_{n+1}(t) = \int_0^t g_n(t-s) g_1(s) ds.$$

If we assume $(*)$ holds for n, then we can just plug into the convolution and we get

$$g_{n+1}(t) = \frac{\lambda^{n+1}}{(n-1)!} e^{-\lambda t} \int_0^t s^{n-1} ds = \lambda \frac{(\lambda t)^n}{n!} e^{-\lambda t}$$

which shows that it holds for $n+1$ as well. By induction it holds for all $n = 1, 2, \ldots$

Problem. What is the probability that no jump occurs in the interval $[0, t]$?

Solution. By the above the probability is $P(N(t) = 0) = e^{-\lambda t}$.

Problem. What is the expected number of jumps in the interval $[0, t]$?

Solution 1. The answer is $E[N(t)] = \sum_{n=0}^{\infty} n \frac{(\lambda t)^n}{n!} e^{-\lambda t} = \lambda t$.

Solution 2. The expected distance between jumps is λ^{-1}, so the expected number of jumps in an interval of length t should be λt.

Lack of memory in the Poisson process. Suppose you have been waiting a while for the Poisson process to make a jump. How much longer do you have to wait? More precisely, what is $P(X_i \geq s + t \mid X_i \geq s)$?

By Bayes rule we get

$$P(X_i \geq s + t \mid X_i \geq s) = \frac{P(X_i \geq s + t)}{P(X_i \geq s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda s}.$$

The conclusion is that the information that you have been waiting already for time s does not change the probability at all, in other words the Poisson process has no memory. Or in other words it is a Markov process.

This is why the Poisson process is often used to model unpredictable things like catastrophic events: earthquakes, hurricanes, or things like arrival times of customers, flips in the orientation of the magnetic pole, etc.

Poisson limit theorem. Let $Y_1^n, Y_2^n, \ldots$ be independent random variables with $P(Y_i^n = 1) = \frac{\lambda}{n}$ and $P(Y_i^n = 0) = 1 - \frac{\lambda}{n}$ and let

$$N^{(n)}(t) = \sum_{i=0}^{\lfloor nt \rfloor} Y_i^n.$$

Then as $n \to \infty$, $N^{(n)}(t)$ converges to the Poisson process with intensity λ.

Proof. All we need to show is that the interarrival times are asymptotically exponentially distributed with mean λ^{-1}. The probability that the interarrival time in $N^{(n)}(\cdot)$ is greater than or equal to t is the probability that about nt random variables in a row are 0, which is

$$\left(1 - \frac{\lambda}{n}\right)^{nt}.$$

Letting $n \to \infty$ we get $e^{-\lambda t}$ as desired.

Sums of independent Poisson processes. Suppose $N_1(t)$ and $N_2(t)$ are independent Poisson processes with intensities λ_1 and λ_2, then $N_1(t) + N_2(t)$ is a Poisson process with intensity $\lambda_1 + \lambda_2$.

Proof. We just have to show that if X_1 and X_2 are independent exponentials of intensity λ_1 and λ_2, then $\min(X_1, X_2)$ is exponential of intensity $\lambda_1 + \lambda_2$. But

$$P(\min(X_1, X_2) \geq t) = P(X_1 \geq t, X_2 \geq t) = P(X_1 \geq t)P(X_2 \geq t) = e^{-\lambda_1 t} e^{-\lambda_2 t} = e^{-(\lambda_1 + \lambda_2)t}$$

as desired.

Compound Poisson process. Let $Y_1, Y_2, \ldots$ be i.i.d. r.v.'s with common distribution $F(x) = P(Y_i \leq x)$. Let $N(t)$ be a Poisson process, then

$$M(t) = Y_1 + \cdots + Y_{N(t)}$$

is called a compound Poisson process. This process waits an exponential amount of time, then jumps a distance Y_1, then waits another exponential amount of time, then jumps a distance Y_2, etc.

The distribution of a compound Poisson with intensity λ at time t is given by

$$P(M(t) \leq x) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} F^{n*}(x)$$

where F^{n*} is the n-fold convolution of F, in other words the distribution of $Y_1 + \cdots + Y_n$.

Proof. $P(M(t) \leq x) = \sum_{n=0}^{\infty} P(M(t) \leq x \mid N(t) = n) P(N(t) = n) = \sum_{n=0}^{\infty} P(Y_1 + \cdots + Y_n \leq x) P(N(t) = n)$.

Example: Winnowing a Poisson process. Suppose Y_i are 1 with probability p and 0 with probability $1 - p$. Then the process $M(t)$ decides to 'accept' each jump with probability p and otherwise ignores it. The probability can be computed above but it is a bit complicated. On the other hand, how long do we have to wait for a jump? Clearly $X_1 + \cdots + X_R$ where R is the first i such that a $Y_i = 1$.

$$P(X_1 + \cdots + X_R \geq x) = \sum_{r=1}^{\infty} P(X_1 + \cdots + X_r \geq x) P(R = r)$$
$$\sum_{r=1}^{\infty} e^{-\lambda x} \left(1 + \frac{\lambda x}{1!} + \cdots + \frac{(\lambda x)^{r-1}}{(r-1)!}\right) p^{r-1}(1-p)$$
$$= e^{-\lambda p x} \left(1 + \frac{\lambda p x}{1!} + \cdots + \frac{(\lambda p x)^{r-1}}{(r-1)!}\right).$$

In other words $M(t)$ is just a Poisson process with intensity λp.

Compound Poisson processes as Markov processes. Because of the lack of memory of the exponential distribution all compound Poisson processes are Markov. If f is any reasonable function then

$$f(M(t)) - f(M(0)) - \int_0^t Lf(M(s))ds$$

is a martingale where the Markov generator L is given by

$$Lf(x) = \lambda \int (f(x+y) - f(x))dF(y).$$

In particular $u(t, x) = E_x[f(M(t))]$ solves the equation $\frac{\partial u}{\partial t} = Lu$ with $u(0, x) = f(x)$.

Problems.

1. Suppose we want to solve the one dimensional stochastic differential equation $dX(t) = \sigma(X(t))dB(t) + b(X(t))dt$. Suppose that we solve first $dY(t) = dB(t) + g(Y(t))dt$ and we try to find a solution $X(t)$ of the form $X(t) = f(Y(t))$ of the original equation. Write down a set of equations that f and g must satisfy for this method to work.
2. Let S_n^p be the sum of n Bernoulli trials (i.i.d. random variables taking values 0 or 1) with probability p to be 1. Suppose we want to find an approximate value for 1. $P(35 \leq S_1^{1/2}00 \leq 45)$; 2. $P(6 \leq S_2^{0.05}0 \leq 12)$. Would you use the central limit theorem or the Poisson limit theorem? Why? Find the appropritate approximate values.
3. In an insurance model for San Francisco, earthquakes over 7 are assumed to happen according to a Poisson process, with expected time between earthquakes of 20 years. We would like to compute the expected waiting time to the next earthquake.

Method I. Because of the lack of memory of the Poisson process the expected waiting time should not depend on when the last earthquake happened, so it must be 20 years.

Method II. The time when we ask the question is chosen at random in the interval between two earthquakes. Since the expected length of such an interval is 20 years, by symmetry the expected waiting time till the next earthquake is 10 years.

Which method is correct and why?

4. Suppose that we have m independent Poisson processes $N_1(t), \ldots, N_m(t)$ with intensities $\lambda_1, ldots, \lambda_m$ representing default triggers and n creditors $\alpha = 1, \ldots, n$. When trigger $N_i(t)$ makes a jump, creditor α defaults with probability $w_{i,\alpha}$. Compute

$$s_\alpha = -\frac{1}{t} \log P(creditor\ \alpha\ does\ not\ default\ before\ time\ t).$$

s_α is called the credit spread (on junior debt with zero recovery).