LECTURE 3

Double stakes until a win. Let $X_1, X_2, \ldots$ be a sequence of Bernoulli trials, i.e. independent with $P(X_i = 1) = p$, $P(X_i = -1) = 1 - p$. Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and suppose that a_n is measurable with respect to $\mathcal{F}_{n-1}$. a_n is the strategy, and your winnings at time n are

$$S_n = \sum_{j=1}^{n} a_j X_j.$$

Clearly if $p = 1/2$ the game is fair and $Y_n = X_1 + \cdots + X_n$ is a martingale. A particular strategy is

$$a_n = \begin{cases} 2^{n-1} & \text{if } X_1 = X_2 = \cdots = X_{n-1} = -1, \\ 0 & \text{otherwise.} \end{cases}$$

The player simply doubles his bet until he wins, and then leaves the table. Note that if he wins on the nth trial, he has lost a total of $1 + 2 + 2^2 + 2^{n-2} = 2^{n-1} - 1$ and gained 2^{n-1} so he is up 1. On the other hand $E[S_n] = 0$ since $S_0 = 0$ and S_n is a martingale.

Stopping times. Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{F}_n$ a filtration. A *stopping time* τ is a non-negative random variable which has the property that

$$\{\omega : \tau(\omega) \leq n\} \in \mathcal{F}_n \qquad \text{for each } n.$$

In words: The decision to stop by time n can be made with only information up to and including that at time n .

Examples. 1. The time when the player wins in the doubling example is a stopping time.

2. Let S_n be a simple random walk. The first time to hit level a,

$$\tau(\omega) = \min\{n : S_n(\omega) = a\}$$

is a stopping time. The last time to hit a,

$$\sigma(\omega) = \max\{n : S_n(\omega) = a\}$$

is *not* a stopping time because you cannot know it is the last without looking into the future.

The *stopped σ-field* is given by

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau = n\} \in \mathcal{F}_n \text{ for all } n\}.$$

$\mathcal{F}_\tau$ is the information you have when you stop.

Theorem (Optional stopping). Let X_n be a martingale with respect to $\mathcal{F}_n$ and τ and σ *bounded* stopping times, i.e. $\sigma, \tau \leq C$ for some $C < \infty$ with $\sigma \leq \tau$ almost surely. Then

$$E[X_\tau \mid \mathcal{F}_\sigma] = X_\sigma \qquad a.s.$$

Proof. Going back to the definition of conditional expectation, what we need to show is that for every $A \in \mathcal{F}_\sigma$,

$$\int_A X_\tau dP = \int_A X_\sigma dP.$$

It would be enough to show that for each ℓ,

$$\int_{A\cap\{\sigma=\ell\}} X_\tau dP = \int_{A\cap\{\sigma=\ell\}} X_\sigma dP$$

because then we could just sum over ℓ and get what we want. Now the right hand side is just

$$\int_{A\cap\{\sigma=\ell\}} X_\ell dP$$

By the martingale property, since $A\cap\{\sigma=\ell\}\in\mathcal{F}_\ell$,

$$\int_{A\cap\{\sigma=\ell\}} X_\ell dP = \int_{A\cap\{\sigma=\ell\}} X_C dP.$$

On the other hand, the left hand side

$$\int_{A\cap\{\sigma=\ell\}} X_\tau dP = \sum_{m=\ell}^{C} \int_{A\cap\{\sigma=\ell\}\cap\{\tau=m\}} X_\tau dP.$$

By the martingale property, since $A\cap\{\sigma=\ell\}\cap\{\tau=m\}\in\mathcal{F}_m$,

$$\sum_{m=\ell}^{C} \int_{A\cap\{\sigma=\ell\}\cap\{\tau=m\}} X_\tau dP = \sum_{m=\ell}^{C} \int_{A\cap\{\sigma=\ell\}\cap\{\tau=m\}} X_C dP.$$

Summing over m we get the same thing on both sides, which completes the proof.

Distribution of hitting times. Let S_n be a simple symmetric random walk starting at 0. Let

$$\tau(\omega) = \min\{n : S_n = a\}.$$

This is called the *hitting time* or the *first passage time*. We will now use the optional stopping theorem to compute the distribution of the random variable τ. The first thing is to note that for any λ,

$$\exp\{\lambda S_n - n\log G(\lambda)\}$$

is a martingale, where $G(\lambda) = E[e^{\lambda X_i}] = (e^\lambda + e^{-\lambda})/2$. Now τ is not a bounded stopping time, but for each fixed $N < \infty$, $\tau\wedge N$ is a bounded stopping time. By the optional stopping time theorem with $\sigma = 0$,

$$E[\exp\{\lambda S_{\tau\wedge N} - \tau\wedge N\log G(\lambda)\}] = 1.$$

Letting $N \to \infty$ by the bounded convergence theorem we get

$$E[\exp\{\lambda a - \tau \log G(\lambda)\}] = 1.$$

Let $h(\gamma)$ be the inverse function of $\log G(\lambda)$ on the positive axis. Then we have

$$E[e^{\gamma\tau}] = e^{h(\gamma)a}.$$

One can check that $P(\tau = n) = 0$ unless n and a have the same parity, in which case

$$P(\tau = n) = \frac{a}{n}\binom{n}{\frac{n+a}{2}}2^{-n}.$$

It is most interesting to study this for a large. Then one can check by the normal approximation that as $a \to \infty$,

$$P(\tau \leq ta^2) \to \sqrt{\frac{2}{\pi}}\int_{t^{-1/2}}^{\infty} e^{-x^2/2}dx.$$

Another simple thing to check (for which you don't have to invert the Laplace transform) is the expected waiting time:

$$E[\tau] = \frac{d}{d\gamma}e^{h(\gamma)a}\Big|_{\gamma=0} = \infty.$$

Although you hit level a with probability 1, the expected time to do so is infinite.

Reflection principle. There is another way to compute the hitting time distribution above. This is the reflection principle, which is roughly the same thing as the method of images for partial differential equations. It says that for simple symmetric random walks

$$P(\max_{1\leq i\leq n} S_i \geq a) = 2P(S_n \geq a).$$

This is because for every path that ends above a we can produce a path that has a maximum above a but ends below a by reflecting the part of the path after the first time that it hits a.

Suppose we want to know the probability that at time $n-1$ the random walk is at $a-1$, and that its maximum up to time $n-1$ is $\leq a-1$. This is just the probability that at time $n-1$ the walk is at $a-1$ minus the probability that at time $n-1$ the walk is at $a-1$ and the maximum up to time $n-1$ is $\geq a$. But by reflection this is just

$$P(X_{n-1} = a-1) - P(X_{n-1} = a+1).$$

So now the probability that $\tau = n$ is just 1/2 this

$$P(\tau = n) = \frac{1}{2}(P(X_{n-1} = a-1) - P(X_{n-1} = a+1)).$$

After a short computation you get the same answer as before.

Misuse of the optional stopping time theorem. Consider the doubling example. Let τ be the first time that you win. From the computation we did we know that $S_\tau = 1$. But S_n is a martingale with $S_0 = 0$ so the optional stopping theorem would say that $E[S_\tau] = 0$. The reason it does not work is that τ is not a bounded stopping time. Introducing $\tau \wedge N$ as in the previous case doesn't work either because although $E[S_{\tau\wedge N}] = 0$ the integrand is not bounded so it is *not* true that $\lim n \to \infty E[S_{\tau\wedge N}] = E[\lim n \to \infty S_{\tau\wedge N}]$.

PROBLEMS.

1. i. Let X be a random variable on $(\Omega, \mathcal{F}, P)$ with $E[|X|] < \infty$. Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}$ be a filtration in $\mathcal{F}$ and define

$$X_n = E[X \mid \mathcal{F}_n].$$

Show that X_n is a martingale with respect to $\mathcal{F}_n$.

ii. Let $Y_1, Y_2, \ldots$ be independent random variables with $P(Y_i = 0) = P(Y_i = 2) = 1/2$. Let $\mathcal{F}_n = \sigma(Y_1, \ldots, Y_n)$ and

$$X_n = \prod_{i=1}^{n} Y_i.$$

Show that X_n is a martingale with respect to $\mathcal{F}_n$, but that there exists *no* integrable random variable X such that

$$X_n = E[X \mid \mathcal{F}_n].$$

2. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables with mean 0 and let $S_n = X_1 + \cdots + X_n$. Show that the sequences

$$M_n^1 = S_n^2 - n\sigma^2, \qquad \sigma^2 = E[X_j^2],$$

$$M_n^2 = e^{\lambda S_n - n\log G(\lambda)}, \qquad G(\lambda) = E[e^{\lambda X_j}],$$

and

$$M_n^3 = e^{izS_n - n\log\phi(z)}, \qquad \phi(z) = E[e^{izX_j}],$$

are martingales with respect to $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

3. i. Show that every martingale $M_n, \mathcal{F}_n$ has *uncorrelated increments*, i.e. if $t_1 < t_2 < t_3 < t_4$ then the two increments $M_{t_4} - M_{t_3}$ and $M_{t_2} - M_{t_1}$ are uncorrelated.

ii. Let M_n and N_n be two martingales with respect to $\mathcal{F}_n$. Show that

$$E[M_n N_n] = \sum_{i=1}^{n} E[(M_n - M_{n-1})(N_n - N_{n-1})] + E[M_0 N_0].$$

3. Suppose $X_1, X_2, \ldots$ are random variables and we know that the joint densities of $(X_1, \ldots, X_n)$ are either $p_n(x_1, \ldots, x_n)$ or $q_n(x_1, \ldots, x_n)$. But we do not know which. To decide, statisticians introduce the *likelyhood ratios*

$$L_n = \frac{q_n(X_1, \ldots, X_n)}{p_n(X_1, \ldots, X_n)}.$$

Lets suppose the densities are positive and continuous so we don't have to worry about anything crazy. The idea is that if p_n are the right densities then the ratio should be very little because the observed values of $X_1, \ldots, X_n$ will tend to be where p_n is large. On the other hand if q_n are right, then the ratio should be very big.

Problem: Show that if p_n are correct, then L_n is a martingale with respect to $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$.

4. An urn contains r red and g green balls. At each time step a ball is drawn at random. It is then replaced in the urn along with c balls of the same colour. Let X_n be the proportion of green balls. Show that X_n is a martingale.
5. (Branching processes) Let $X_1, X_2, \ldots$ be a sequence of non-negative integer valued random variables which have the property that, given $X_1, \ldots, X_n$, the variable X_{n+1} is a sum $Y_1^n + \cdots + Y_{X_n}^n$ of random variables, again taking only integer values. You can think of X_n individuals in the nth generation and the ith individual gives birth to Y_i^n children. Assume the whole array Y_i^n are independent and indentically distributed.
Problem. Find a normalization z_n so that X_n/z_n is a martingale.