LECTURE 6

Continuity of Brownian motion. With probability one, the sample path $\beta_t(\omega)$ is a continuous but nowhere differentiable function. First we prove that it is continuous. We need the following useful estimate on the tails of normal distributions.

Lemma. If X is $\mathcal{N}(0,1)$ then $P(|X| \geq a) \leq e^{-a^2/2}$.

Proof. We use the exponential Tchebyshev's inequality: For $\lambda > 0$,

$$P(|X| \geq a) \leq E[e^{\lambda X}]/e^{\lambda a}.$$

Recalling the Laplace transform of the normal distribution, the right hand side is $e^{-\lambda a+\lambda^2/2}$. To get the lemma, just choose $\lambda = a$.

It is also useful to introduce the Haar basis of L^2,

$$\mathcal{H}_{m,j}(x) = 2^{\frac{m+1}{2}} \left\{ \mathbf{1}_{[\frac{2j}{2^{m+1}}, \frac{2j+1}{2^{m+1}})} - \mathbf{1}_{[\frac{2j+1}{2^{m+1}}, \frac{2j+2}{2^{m+1}})} \right\}$$

and their integrals the Schauder functions

$$\mathcal{S}_{m,j}(x) = \int_0^x \mathcal{H}_{m,j}(y)dy.$$

$\mathcal{S}_{m,j}(x)$ is a tent of height $2^{-\frac{m+1}{2}}$ between $\frac{2j}{2^{m+1}}$ and $\frac{2j+2}{2^{m+1}}$.

Theorem. Brownian motion has continuous sample paths.

Proof. Let $X(t)$ be Brownian motion on $[0,1]$ and let $X_n(t)$ be the polygonal approximation of $X(t)$ on the points $0, 1/2^n, 2/2^n, \dots$ i.e. $X_n = X$ at those points and X_n is just linear in between. Clearly $X_n(t,\omega)$ is a continuous function for each ω and $X_n(t,\omega) \to X(t,\omega)$ as $n \to \infty$ for each ω. If we can show that the convergence is uniform for $0 \leq t \leq 1$ then the limit $X(t,\omega)$ is a continuous function. So our job is to show this for almost every ω. Now $X_n(t) - X(t) = \sum_{m=n}^{\infty} X_{m+1}(t) - X_m(t)$ so $\sup_{0\leq t\leq 1} |X_n(t) - X(t)| \leq \sum_{m=n}^{\infty} \sup_{0\leq t\leq 1} |X_{m+1}(t) - X_m(t)|$ so it is enough to show that, for each $\epsilon > 0$, with probability one, there is a $N(\omega)$ such that for all $n \geq N(\omega)$,

$$\sum_{m=n}^{\infty} \sup_{0\leq t\leq 1} |X_{m+1}(t) - X_m(t)| \leq \epsilon. \qquad (*)$$

Now note that

$$X_{m+1}(t) - X_m(t) = \sum_{j=0}^{2^m-1} \xi_{j,m}\mathcal{S}_{j,m}(x)$$

where $\mathcal{S}_{j,m}$ are Schauder functions and $\xi_{j,m}$ is a Gaussian family. It is not hard to check that the $\xi_{j,m}$ have mean 0, variance 1 and are independent. It is also clear that

$$\sup_{0\leq t\leq 1} |X_{m+1}(t) - X_m(t)| = 2^{-\frac{m+1}{2}} \max_{0\leq j\leq 2^m-1} |\xi_{j,m}|.$$

So $(*)$ becomes: For each $\epsilon > 0$, with probability one, there is a $N(\omega)$ such that for all $n \geq N(\omega)$,

$$\sum_{m=n}^{\infty} 2^{-\frac{m+1}{2}} \max_{0 \leq j \leq 2^m - 1} |\xi_{j,m}| \quad \leq \epsilon \tag{$**$}$$

Now

$$P(2^{-\frac{m+1}{2}} \max_{0 \leq j \leq 2^m-1} |\xi_{j,m}| \geq 2^{-m/4}) \leq 2^m P(2^{-\frac{m+1}{2}} |\xi_{0,m}| \geq 2^{-m/4}).$$

Applying the lemma, we find it is summable in m. So

$$\sum_{n=1}^{\infty} P(\sum_{m=n}^{\infty} 2^{-\frac{m+1}{2}} \max_{0 \leq j \leq 2^m-1} |\xi_{j,m}| \geq \sum_{m=n}^{\infty} 2^{-m/4}) < \infty$$

so by Borel-Cantelli lemma, with probability one,

$$\sum_{m=n}^{\infty} 2^{-\frac{m+1}{2}} \max_{0 \leq j \leq 2^m-1} |\xi_{j,m}| \leq \sum_{m=n}^{\infty} 2^{-m/4}$$

for all $n \geq N_1(\omega)$ for some finite $N_1(\omega)$. But for any ϵ the right hand side is less than ϵ for all $n \geq N_2$. Taking $N(\omega) = \max(N_1, N_2)$ we have that for any $n \geq N(\omega)$ $(**)$ holds, which is what was to be proved.

The sample paths of Brownian motion are nowhere differentiable functions with probability one. It is very important to remember this and to see that expressions such as the stochastic differential equation

$$dx = \sigma d\beta + bdt$$

have no classical calculus meaning because of this. The non-differentiability is a bit hard to prove, so we prove something similar, and actually more to the point for our purposes of stochastic calculus. If $F(t)$ is a function on $[0,1]$ one can define its total variation by

$$\|F\|_{TV} = \lim_{n \to \infty} \sum_{j=0}^{2^n-1} |F(\frac{j+1}{2^n}) - F(\frac{j}{2^n})|.$$

If $\|F\|_{TV}$ is finite we say F has bounded variation. It turns out that F has bounded variation if and only if $\mu((a,b]) = F(b) - F(a)$ generates a (signed) measure. Certainly if F is differentiable with $F' = f$ then $\mu(A) = \int_A f(x)dx$ is a nice measure, so if F doesn't have bounded variation it must be very non-differentiable since its 'derivative' isn't even a measure. Now suppose we define the *second variation* of a function

$$\|F\|_2 = \lim_{n \to \infty} \sum_{j=0}^{2^n-1} |F(\frac{j+1}{2^n}) - F(\frac{j}{2^n})|^2.$$

If $\|F\|_2$ is finite then we say F has finite second variation. If a function is continuous and has finite second variation, then it cannot be of bounded variation.

Theorem. Brownian motion paths have finite second variation with probability one.

Proof. Lets call $\xi_{n,j} = \beta(\frac{j+1}{2^n}) - \beta(\frac{j}{2^n})$. For each n they are independent for $j = 0, \dots, 2^n - 1$ and they are $\mathcal{N}(0, 2^{-n})$. We have $Y_n = \sum_{j=0}^{2^n-1} \xi_{n,j}^2$ is mean 1 and we want to show that it converges to 1 almost surely. One can compute $E[(Y_n - 1)^2] = 2^n Var(\xi_{n,0}^2)$. This is just $C2^{-n}$. By Tchebyshev, $P(|Y_n - 1| \geq \epsilon) \leq C2^{-n}\epsilon^{-1}$. So $P(|Y_n - 1| \geq \epsilon)$ are summable and therefore, by Borel-Cantelli, with probability one, there exists an $N(\omega)$ such that for all $n \geq N(\omega)$, $|Y_n - 1| < \epsilon$. This completes the proof.

Martingale property of Brownian motion. A continuous time process $X_t,\ t \geq 0$ is called a martingale with respect to an increasing family of σ-fields $\mathcal{F}_t$ (a filtration) if for any $s < t$,

$$E[X_t \mid \mathcal{F}_s] = X_s, \qquad a.s.$$

Brownian motion $\beta_t,\ t \geq 0$ is a martingale with respect to its *canonical filtration*, $\mathcal{F}_t = \sigma(\beta_s\ , 0 \leq s \leq t)$ since $\beta_t - \beta_s$ is mean zero and independent of $\mathcal{F}_s$.

Exponential martingales. $\beta_t,\ t \geq 0$ is Brownian motion if and only if for each λ,

$$Z_\lambda(t) = e^{\lambda\beta_t - \frac{1}{2}\lambda^2 t}$$

is a martingale.

Proof. Suppose $\beta_t,\ t \geq 0$ is Brownian motion. Then $Z_\lambda(t)/Z_\lambda(s) = e^{\lambda[\beta_t - \beta_s] - \frac{1}{2}\lambda^2[t-s]}$ is independent of $\mathcal{F}_s$ with expectation 1 since $\beta_t - \beta_s$ is normal with mean 0 and variance $t - s$ (just because the Laplace transform of a $\mathcal{N}(m, \sigma^2)$ is $e^{m\lambda + \sigma^2\lambda^2}$). This proves that

$$E[Z_\lambda(t)/Z_\lambda(s) \mid \mathcal{F}_s] = 1.$$

But $Z_\lambda(s)$ is measurable with respect to $\mathcal{F}_s$ so it can come out of the conditional expectation and you get the martingale property.

On the other hand suppose that all those exponentials are martingales. The martingale property shows that $Z_\lambda(s)$ is measurable with respect to $\mathcal{F}_s$. Then the martingale property gives

$$E[e^{\lambda[\beta_t - \beta_s] - \frac{1}{2}\lambda^2[t-s]} \mid \mathcal{F}_s] = 1$$

for any λ and $t > s$. This tells us that conditioned on $\mathcal{F}_s$, $\beta_t - \beta_s$ is $\mathcal{N}(0, t - s)$. But also the conditional expectation is not even a random variable! By approximation we can check that

$$E[g(\beta_t - \beta_s) \mid \mathcal{F}_s] = \int g(x) \frac{e^{-\frac{x^2}{2(t-s)}}}{\sqrt{2\pi(t-s)}} dx$$

the point being that it is independent of ω. We conclude that $\beta_t - \beta_s$ is $\mathcal{N}(0, t - s)$ and independent of $\mathcal{F}_s$. So $\beta_t,\ t \geq 0$ is Brownian motion.

Optional stopping (continuous time version). A non-negative random variable is a stopping time with respect to a filtration $\mathcal{F}_t,\ t \geq 0$ if for each $t \geq 0$,

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t.$$

It means that by time t you have the information necessary to know if have stopped by time t.

The continuous time version of the optional stopping theorem says that if τ and σ are two bounded stopping times, with $\sigma \leq \tau$ then

$$E[X_\tau \mid \mathcal{F}_\sigma] = X_\sigma.$$

Bounded here means $\tau \leq C$ for some finite C almost surely, and the theorem may very well not hold if the stopping times are not bounded. For example, let β_t be a one-dimensional Brownian motion starting at 0 and let τ be the first time that it gets to 1. Note that $\tau < \infty$ almost surely, because the process is recurrent. But $X_\tau = 1$ always. If we let $\sigma = 0$ then the optional stopping time theorem would tell us

$$1 = E[X_\tau \mid \mathcal{F}_0] = X_0 = 0$$

which is clearly false. The reason it didn't work is that although $\tau < \infty$ almost surely, it is not true that there is a finite constant C with $\tau \leq C$ almost surely. You know that a Brownian motion starting at 0 will get to 1 eventually, but it make take a long time.

Distribution of the hitting time (first passage time) for Brownian motion. Let τ be the first time that Brownian motion hits 1. We will use the optional stopping time theorem to compute the distribution of τ. Note that we cannot use it directly because τ is unbounded. So we replace τ by $\tau_n = \min(\tau, n)$. By the optional stopping time theorem with $\sigma = 0$ (an excellent time to stop),

$$E[e^{\lambda \beta_{\tau_n} - \frac{1}{2}\lambda^2 \tau_n}] = 1.$$

We want to take the limit as $n \to \infty$ inside the integral. If we write it as $E[f_n]$ we want $\lim_{n\to\infty} E[f_n] = E[\lim_{n\to\infty} f_n]$. Recall the bounded convergence theorem from analysis which says that this is ok if there is some $C < \infty$ such that $|f_n(\omega)| \leq C$ for all n and all ω. In our case $|f_n| \leq e^\lambda$ because $\beta_{\tau_n} \leq 1$. But as $n \to \infty$ we have $\tau_n \to \tau$ and $\beta_\tau = 1$ so we get

$$E[e^{\lambda - \frac{1}{2}\lambda^2 \tau}] = 1.$$

Changing variables $\alpha = \frac{1}{2}\lambda^2$ this becomes

$$E[e^{-\alpha\tau}] = e^{-\sqrt{2\alpha}}.$$

So we computed the Laplace transform of the distribution of τ. Now you look in a book of Laplace transforms and find out that this is the Laplace transform of $e^{-\frac{1}{2t}}/2\pi t^3$. So

$$P(\tau \leq t) = \int_0^t \frac{e^{-\frac{1}{2s}}}{2\pi s^3} ds.$$

Now let τ_a be the hitting time of a instead of 1. We could go through the same argument or instead we could use the rescaling property of Brownian motion that $\tilde{\beta}_t =$

$a\beta_{t/a^2}$ is another Brownian motion. The time $\tilde{\beta}$ hits a is a^2 times the time that β hits 1. So the distribution of τ_a is the same as the distribution of $a^2\tau$.

$$P(\tau_a \le t) = P(\tau \le a^2 t) = \int_0^{a^2} t\frac{e^{-\frac{1}{2s}}}{2\pi s^3}ds = \int_0^t \frac{ae^{-\frac{a}{2u}}}{2\pi u^3}du.$$

Another way to compute the distribution of the hitting time is with the *reflection principle* which says that

$$P(\sup_{0\le s\le t} \beta_s \ge a) = 2P(\beta_t \ge a).$$

Note that the left hand side is just the probability that τ_a is less than t and the right hand side is just a Gaussian. So it translates to

$$P(\tau_a \le t) = 2\int_a^\infty \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}dx.$$

Letting $u = at/x^2$ you get the same formula as above. Although this method with the reflection principle is very simple it is a little too dependent on special symmetry properties of Brownian motion.

Different characterizations of Brownian motion.

1. Brownian motion is the Markov process with transition probabilities $p(s,x,t,dy) = \exp -|x-y|^2/2(t-s)/(2\pi(t-s))^{d/2}$.
2. Brownian motion is the Gaussian process with mean 0 and covariance $\min(t,s)$.
3. Brownian motion is a process with continuous trajectories and stationary independent increments.
4. Browian motion is the process such that

$$Z_\lambda(t) = e^{\lambda\beta_t - \frac{1}{2}\lambda^2 t}$$

are martingales for each λ. It is also enough if

$$Y_f(t) = f(\beta_t) - \frac{1}{2}\int_0^t f''(\beta_s)ds$$

are martingales for every smooth function f.

PROBLEMS.

1. Let β_t, $t \ge 0$ be Brownian motion starting at 0 and let τ be the first time that it hits $\pm a$, i.e.

$$\tau = \inf\{t \ge 0 \ : \ |\beta_t| \ge a\}.$$

   i. Show that $\beta_t^2 - t$ is a martingale and use this to compute $E[\tau]$.
   ii. Find the distribution of τ (or at least find its Laplace transform) and use this to compute the moments of τ.
   iii. Let $\sigma = \inf\{t \ge 0 \ : \ \beta_t \ge a\}$. Find $E[\sigma]$.

2. Let β_t, $t \geq 0$ be Brownian motion. Use the fact that for any nice function f, $Y_t = f(\beta_t) - \frac{1}{2}\int_0^t f''(\beta_s)ds$ is a martingale in t to show that e^{β_t} is not a martingale but $e^{\beta_t - t/2}$ is a martingale. (In stochastic calculus the former is the solution of $d\ln(X) = d\beta$ and the latter is the solution of $dX = Xd\beta$.)
3. Let $X_n(t)$ be the polygonalization of Brownian motion defined in the proof of continuity. Let $Y_n(t) = \frac{d}{dt}X_n(t)$
   i. Show that $Y_n(t)$ has no limit as $n \to \infty$.
   ii. Compute the mean $m_n(t) = E[Y_n(t)]$ and covariance $\rho_n(t-s) = E[Y_n(t)Y_n(s)]$.
   iii. Compute the spectral density μ_n defined by $\rho_n(t) = \int_{-\infty}^{\infty} e^{it\lambda}d\mu_n(\lambda)$ and find the limit $\mu = \lim_{n\to\infty}\mu_n$.
   iv. Explain why the mythical $Y = \lim_{n\to\infty} Y_n$ is known as *white noise.*
4. Let $X_1, X_2, \ldots$ be a sequence of random variable with mean 0 and covariance $E[X_iX_j] = 1$ if $i = j$ and 0 otherwise. Let $a_0, a_2, a_3, \ldots$ be a sequence of numbers.
   i. Fix m and for each n let Y_n be the moving average $Y_n = a_0X_n + \cdots + a_mX_{n-m}$. Find the covariance $\rho(n-m) = Cov(Y_m, Y_n)$ and compute the spectral density μ given by $\rho(n) = \int_{-\pi}^{\pi} e^{in\lambda}d\mu(\lambda)$.
   ii. Same as in i. except for Z_n defined by

$$a_0Z_n + \cdots + a_mZ_{n-m} = X_n.$$

5. Let $X_0, X_1, \ldots$ be a simple random walk on the integers, starting at 0. In other words $X_0 = 0$ and $X_{i+1} - X_i$ is $+1$ with probability p and -1 with probability $1-p$. Let $a < 0$ and $b > 0$ and let

$$\tau = \min\{n \ : \ X_n = a \text{ or } b\}.$$

   i. Compute $P(S_\tau = a)$.
   ii. Compute $E[\tau]$.