LECTURE 8

Today we are going to have an informal lecture on the connection between probability and partial differential equations. It will also give us a chance to introduce some of the most basic pde's.

Dirichlet problem. Let D be a domain in $\mathbf{R}^d$, i.e. a bounded connected open set with a smooth boundary ∂D. You are presented with a continuous function f defined on the boundary ∂D and you are asked to come up with a function u satisfying

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = f & \text{on } \partial D. \end{cases}$$

Here the Laplacian Δ is the operator

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2}.$$

Discrete version of the Dirichlet problem. As a first try we approximate the problem on a lattice, with lattice width ϵ taken pretty small. Let D_ϵ be the approximation of D and ∂D_ϵ the boundary. The discrete Laplacian is given by

$$\Delta_\epsilon u(x) = \epsilon^{-2} \sum_{y \sim x} [u(y) - u(x)]$$

where $y \sim x$ means that y is a nearest neighbour lattice site to x. We have a function f defined on the lattice sites in the boundary and the discrete Dirichlet problem is to come up with a function u defined on D_ϵ satisfying

$$\begin{cases} \Delta_\epsilon u = f & \text{in } D_\epsilon, \\ u = f & \text{on } \partial D_\epsilon. \end{cases}$$

Solution by random walks. Let X_n be a symmetric simple random walk starting at $x \in D_\epsilon$. In other words

$$X_n = Y_1 + \cdots + Y_n + x$$

where each Y_i is chosen with uniform distribution from the $2d$ basic vectors in the coordinate directions (we use e to denote such basic lattice vectors.) Let τ be the first time that the random walk is in the boundary ∂D_ϵ. Then X_τ is the first site in the boundary that the walk gets to. We claim the answer to our problem is

$$u(x) = E_x[f(X_\tau)].$$

The subscript x tells you that the walk starts at x.

We have two things to check. First that $\Delta_\epsilon u = 0$ inside. By double conditioning we have

$$u(x) = E_x[E_x[f(X_\tau) \mid Y_1]].$$

This latter can be written as

$$\sum_e E_x[f(X_\tau) \mid Y_1 = e]P(Y_1 = e).$$

The last term $P(Y_1 = e) = 1/2d$. Note that

$$E_x[f(X_\tau) \mid Y_1 = e] = E_{x+e}[f(X_\tau)] = u(x+e).$$

So we have that

$$u(x) = Av_{y \sim x} u(y)$$

and therefore $\Delta_\epsilon u = 0$.

The second thing we have to check is that $u = f$ on ∂D_ϵ. This is obvious since if $x \in \partial D_\epsilon$, then $\tau = 0$ and $X_\tau = x$, so there is no probability and $u(x) = E_x[f(X_\tau)] = f(x)$.

Solution of the Dirichlet problem by Brownian motion. Let W_t be a Brownian motion starting at $x \in D$. Let τ be the exit time

$$\tau = \inf\{t \geq 0 \ : \ W_t \notin D\}.$$

Note that W_τ is the point in the boundary where the Brownian motion exits. We claim the solution of the Dirichlet problem is

$$u(x) = E_x[f(W_\tau)].$$

We need the *mean value property* of harmonic functions which says that

$$\Delta u = 0 \text{ in } D \quad \Leftrightarrow \quad Av_{|y-x|=\epsilon} u(y) = u(x) \text{ for any } \epsilon \text{ with } \{|y-x| \leq \epsilon\} \subset D.$$

The first thing to show is that our u satisfies the mean value property, and is therefore harmonic. Let x be a point in D and ϵ be small enough that the ball of radius ϵ around x is contained in D as well. Let σ denote the hitting time of the surface of the ball of radius ϵ by a Brownian motion starting at x. Then W_σ is the point where the Brownian motion first exits the ball of radius ϵ. By double conditioning we have

$$u(x) = E_x[E_x[f(W_\tau) \mid W_\sigma]].$$

This can be rewritten as

$$\int_{|y-x|=\epsilon} E_x[f(W_\tau) \mid W_\sigma = y]P(W_\sigma \in dy).$$

The distribution of W_σ is uniform on the sphere of radius ϵ, by the spherical symmetry of Brownian motion. By the strong Markov property the integrand is

$$E_x[f(W_\tau) \mid W_\sigma = y] = E_y[f(W_\tau)] = u(y).$$

Therefore we have proved that $u(x)$ is equal to its average over the sphere of radius ϵ. This proves the mean value property.

The second thing to show is that if x_n is a sequence of points in D with $x_n \to y$ where y is a point in the boundary then $u(x_n) \to f(y)$. Let's write it out. We need

$$E_{x_n}[f(W_\tau)] \to f(y).$$

What it means is that the distribution of the exit point W_τ is becoming concentrated near y. Note that this is non-trivial, and in fact if the boundary is terrible enough (eg. a very sharp inward pointing spike) may not even be correct. The proof is a little tricky so we will leave it for later.

2nd order linear partial differential operators. Let σ be a $d \times d$ matrix, $a = \sigma^T \sigma$, and b a d-vector. Let Y be a random variable with mean 0 and covariance I. Then $Z = \epsilon^{1/2}\sigma Y + \epsilon b$ has mean ϵb and covariance ϵa. Let u be a smooth function on $\mathbf{R}^d$. By Taylor's theorem we have

$$u(x+Z) - u(x) = \sum_{i=1}^{d} Z_i \frac{\partial u}{\partial x_i}(x) + \frac{1}{2} \sum_{i,j=1}^{d} Z_i Z_j \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + O(|Z|^3).$$

So

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} E[u(x+Z) - u(x)] = Lu(x)$$

where

$$Lu = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i}.$$

Generalized Dirichlet problem. Suppose we want to solve

$$\begin{cases} Lu = 0 & \text{in } D, \\ u = f & \text{on } \partial D \end{cases}$$

where

$$Lu = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x) \frac{\partial u}{\partial x_i}.$$

We claim the solution has the following representation: Let X_t be the solution of the stochastic differential equation

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt, \qquad X_0 = x.$$

Let τ be the exit time from the region D. Then X_τ is the exit point on the boundary ∂D. The formula for u is

$$u(x) = E_x[f(X_\tau)].$$

To show it we cannot use the same conditioning argument as before because we lost the symmetry. Suppose u is the solution of the generalized Dirichlet problem. We claim that

$u(X_t)$ is a martingale (note we stop the process when it hits the boundary). To show this we apply Ito's formula to $u(X_t)$. We get

$$du(X) = \sum_{i=1}^{d} \frac{\partial u}{\partial x_i} dX_i + \frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 u}{\partial x_i \partial x_j} dX_i dX_j.$$

To simplify this, plug in the sde. Note that

$$dX_i dX_j = \sum_{\ell,\ell'} \sigma_{i\ell}\sigma_{j\ell'} dW_\ell dW_{\ell'} = \sum_{\ell} \sigma_{i\ell}\sigma_{j\ell} dt = a_{ij} dt.$$

You get

$$du(X) = Lu(X)dt + \sigma(X)\nabla u(X) \cdot dW.$$

In particular, if $Lu = 0$, then $u(X_t) - u(X_0) = \int_0^t \sigma(X_s)\nabla u(X_s) \cdot dW_s$ which is always a martingale. By the optional stopping theorem

$$E_x[u(X_\tau)] = u(x).$$

But note that $X_\tau \in \partial D$, so $u(X_\tau) = f(X_\tau)$ so we are done.

Poisson equation. Suppose we want instead

$$\begin{cases} Lu = f & \text{in } D, \\ u = 0 & \text{on } \partial D \end{cases}$$

where f is some given function defined in D. Let X_t and τ be the same as above. By Ito's formula we have $du(X) = f(X)dt + dM$ where M is a martingale. By the optional stopping time theorem

$$E_x[u(X_\tau) - u(X_0) - \int_0^\tau f(X_s)ds] = 0.$$

But $X_\tau \in \partial D$ so $u(X_\tau) = 0$. Also $u(X_0) = u(x)$ so we have the representation formula

$$u(x) = E_x[\int_0^\tau f(X_s)ds].$$

Heat equation. The heat equation on $\mathbf{R}^d$ is

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u, & t > 0 \\ u(0,x) = f(x). \end{cases}$$

This is one of those special cases where the solution can be written explicitly in terms of functions we have names for. The answer is

$$u(t,x) = \int f(y)p(0,x,t,y)dy$$

where

$$p(0,x,t,y) = \frac{e^{-\frac{|y-x|^2}{2t}}}{(2\pi t)^{d/2}}.$$

Since these are the transition probabilities for Brownian motion we have automatically the representation formula

$$u(t,x) = E_x[f(W_t)].$$

Diffusion equation. In an inhomogeneous medium with convection the heat equation becomes

$$\begin{cases} \frac{\partial u}{\partial t} = Lu, & t > 0 \\ u(0,x) = f(x), \end{cases}$$

where

$$Lu = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(x,t)\frac{\partial u}{\partial x_i}.$$

Let X_t be the solution of the stochastic differential equation

$$dX_t = \sigma(X_t,t)dW_t + b(X_t,t)dt, \qquad X_0 = x.$$

By Ito's formula

$$du(T-t,X_t) = \left\{-\frac{\partial u}{\partial t} + Lu\right\}(T-t,X_t)dt + \sigma(X_t,t)\nabla u(X_t,t)dW_t.$$

So if u solves the diffusion equation then $u(T-t,X_t)$ is a martingale. Therefore

$$E_x[u(T-T,X_T)] = u(T,x).$$

But $u(0,X_T) = f(X_T)$ so we have the formula

$$u(T,x) = E_x[f(X_T)].$$

Diffusion equation with killing (interest). Let r be a constant. We want to solve

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}Lu - ru, & t > 0 \\ u(0,x) = f(x). \end{cases}$$

We can rewrite it as $\frac{\partial e^{rt}u}{\partial t} = \frac{1}{2}\Delta e^{rt}u$ so the answer is just

$$u(t,x) = E_x[e^{-rt}f(X_t)].$$

This is used a lot in finance. For example in the Black-Scholes analysis we want to evaluate

$$u(t,x) = E_x[e^{-rt}(X_t - a)_+]$$

where r is the interest rate, a is the strike price, and X_t is the solution of the sde

$$dX_t = rX_t dt + \sigma X_t dW_t.$$

Now we know that u is the solution of the pde

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru \\ u(0,x) = (x-a)_+. \end{cases}$$

As chance would have it, the explicit answer can be written down in terms of the error function $\Phi(x) = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$. The answer is

$$u(t,x) = x\Phi\left(\frac{1}{\sigma\sqrt{t}}\left\{\log(x/a) + (r+\sigma^2/2)t\right\}\right) + ae^{-rt}\Phi\left(\frac{1}{\sigma\sqrt{t}}\left\{\log(x/a) + (r-\sigma^2/2)t\right\}\right).$$

Feynman-Kac formula. Suppose $V(x)$ is a function and we want to solve

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + V(x)u, \qquad t > 0 \\ u(0,x) = f(x). \end{cases}$$

It's like the heat equation with killing, but the rate is no longer a constant. The Feynman-Kac formula says the answer has the representation

$$u(t,x) = E_x[e^{\int_0^t V(W_s)ds} f(W_t)].$$

To see why this is so, using Ito's formula we have

$$\begin{aligned} d\left\{u(T-t,W_t)e^{\int_0^t V(W_s)ds}\right\} &= \left\{-\frac{\partial u}{\partial t} + \frac{1}{2}\Delta u + Vu\right\}(T-t,X_t)e^{\int_0^t V(W_s)ds}dt \\ &\quad + \sigma(X_t,t)\nabla u(X_t,t)e^{\int_0^t V(W_s)ds}dW_t. \end{aligned}$$

So $u(T-t,W_t)e^{\int_0^t V(W_s)ds}$ is a martingale. Therefore

$$E_x[u(T-T,W_t)e^{\int_0^T V(W_s)ds}] = u(T,x).$$

Since $u(0,W_t) = f(W_t)$ this is the Feynman-Kac formula.

Question. Note that in the above calculation there is no extra Ito term corresponding to the exponential $e^{\int_0^t V(W_s)ds}$. Why not?