LECTURE 9

Stochastic integrals. Suppose $B(t)$ represents the price of a share of an asset at time t, and $\delta(t)$ is the number of shares you hold at time t. The stochastic integral $I(t) = \int_0^t \delta(u)dB(u)$ will represent your wealth at time t.

Unless you are allowed to peek into the future to determine your betting strategy, it makes sense to assume that $\delta(t)$ is *adapted* to the filtration of $B(t)$, i.e. $\delta(t)$ is measurable with respect to $\mathcal{F}(t)$, the σ-field generated by the random variables $B(s)$, $0 \leq s \leq t$.

As a first try, let's suppose that you only make trades at times $0 = t_0 \leq t_1 \leq \cdots \leq t_n = T$. In other words the random variables $\delta(t)$ are constant on intervals $t_i \leq t < t_{i+1}$. For each ω, it is just a step function taking jumps at times $t_1, t_2, \ldots$ The times are not random, just the step sizes (= moves in your portfolio). Then your wealth at time t is clearly

$$I(t) = \sum_{i=0}^{n} \delta(t_i)(B(t_{i+1} \wedge t) - B(t_i \wedge t)).$$

If $B(t)$ is a martingale then this is nothing but the martingale transform. Then (cf. Class 2 notes) we know that $I(t)$ is a martingale as well. Furthermore we know how to compute the variance of a martingale transform. If $B(t)$ is Brownian motion we get

$$E[|I(t)|^2] = E[\int_0^t |\delta(u)|^2 du]. \tag{$*$}$$

Now suppose we have a general $\delta(t)$, i.e. we trade in continuous time. We want to define $I(t) = \int_0^t \delta(u)dB(u)$ in that case. The idea is to choose a sequence $\delta_n(t)$ of discrete time trading strategies, which approximates our continuous time trading strategy in the sense that

$$E[\int_0^T |\delta_n(u) - \delta(u)|^2 du] \to 0 \qquad \text{as } n \to \infty.$$

Clearly for this to make sense we will need

$$E[\int_0^T |\delta(u)|^2 du] < \infty.$$

We always insist that $\delta(t)$ and all the $\delta_n(t)$ are adapted to $\mathcal{F}(t)$. (It is not quite obvious that such an approximating sequence can be chosen. A proof is in the appendix.) Now we can define the stochastic integral $I(t) = \int_0^t \delta(u)dB(u)$ as the limit of $I_n(t) = \int_0^t \delta_n(u)dB(u)$ in the following way. If I and J are random variables we can denote

$$\langle I, J \rangle = E[IJ].$$

If $\delta(t)$ and $\gamma(t)$ are processes we can similarly define

$$\langle\langle \delta, \gamma \rangle\rangle = E[\int_0^T \delta(u)\gamma(u)du].$$

Both of these define Hilbert spaces which we call $L^2(P)$ and $L^2([0,T]\times P)$ respectively. Then $(*)$ says that the map $L^2([0,T]\times P)\to L^2(P)$ given by

$$\delta(\cdot)\mapsto I=\int_0^T \delta(u)dB(u)$$

is an *isometry* between the spaces. Since δ_n is a Cauchy sequence in $L^2([0,T]\times P)$, its image $I_n=\int_0^T \delta_n(u)dB(u)$ is therefore a Cauchy sequence in $L^2(P)$. Since $L^2(P)$ is complete there therefore exists a unique limit, which we call $I=\int_0^T \delta(u)dB(u)$.

Basic properties of the stochastic integral.

1. $I(t)=\int_0^t \delta(u)dB(u)$ is a martingale.
2. $I(t)=\int_0^t \delta(u)dB(u)$ is continuous almost surely.
3. (linearity) $\int_0^t [c_1\delta_1(u)+c_2\delta_2(u)]dB(u)=c_1\int_0^t \delta_1(u)dB(u)+c_2\int_0^t \delta_2(u)dB(u)$.
4. (L^2 isometry) $E[(\int_0^t \delta(u)dB(u))^2]=E[\int_0^t(\delta(u))^2du]$.

$I(t)=\int_0^t \delta(u)dB(u)$ is a stochastic process and to compute it is convenient to know some associated martingales:

5. $(\int_0^t \delta(u)dB(u))^2-\int_0^t(\delta(u))^2du$ is a martingale. In finance lingo, the *instantaneous absolute volatility* of $I(t)$ is $\delta^2(t)$.
6. $e^{\{\int_0^t \delta(u)dB(u)-\frac{1}{2}\int_0^t(\delta(u))^2du\}}$ is a martingale.

Recapitulation of the definition of the stochastic integral. Let $\delta(t)$ be adapted to the Brownian filtration. If δ is continuous and bounded we can just define

$$I_n(t)=\sum_{i=0}^{\infty}\chi_{[0,t]}\delta(\tfrac{i}{n})(B(\tfrac{i+1}{n})-B(\tfrac{i}{n})).$$

Then we use the L^2 isometry and the completeness of $L^2(P)$ to see that the sequence $I_n(t)$ must have a limit in the L^2 sense, an abstract object we name $I(t)$.

Computation of a stochastic integral. The classic example is $\int_0^t B(s)dB(s)$. The approximating sequence is

$$I_n(t)=\sum_{i=0}^{nt-1}B(\tfrac{i}{n})(B(\tfrac{i+1}{n})-B(\tfrac{i}{n})).$$

It is not clear what to do with this. But there is a trick! Let

$$J_n(t)=\sum_{i=0}^{nt-1}B(\tfrac{i+1}{n})(B(\tfrac{i+1}{n})-B(\tfrac{i}{n})).$$

Then

$$J_n(t)-I_n(t)=\sum_{i=0}^{nt-1}(B(\tfrac{i+1}{n})-B(\tfrac{i}{n}))^2\to t.$$

and

$$J_n(t)+I_n(t)=\sum_{i=0}^{nt-1} B^2(\tfrac{i+1}{n})-B^2(\tfrac{i}{n})=B^2(t).$$

So $I_n(t)\to\frac{1}{2}B^2(t)-\frac{1}{2}t=\int_0^t B(s)dB(s)$. There are a couple of morals to this story. First, if we use the right endpoint rule with our approximations, as in $J_n(t)$, we get a different answer. $J_n(t)$ converges to $\frac{1}{2}B^2(t)+\frac{1}{2}t$, the difference being t, the quadratic variation. Nothing like this can happen with standard Riemann integrals because they only deal with functions of finite variation, so the quadratic variation terms just go away. The second moral is that integration by parts cannot follow the same rule for stochastic integrals as for normal integrals because then we would have $\int_0^t B(s)dB(s)=B^2(t)-\int_0^t B(s)dB(s)$ which, rearranges to $\int_0^t B(s)dB(s)=\frac{1}{2}B^2(s)$ which we just saw is false. Same thing with the chain rule. If we let $f(x)=x^2$ then we expect $B^2(t)=f(B(t))=\int_0^t df(B(s))=\int_0^t f'(B(s))dB(s)=\int_0^t 2B(s)dB(s)$ and we get the same wrong expression. So you have to be a little careful to only use rules you prove and not let any intuition from normal integrals slip in and lead you astray. The correct chain rule (Ito's formula) is given below.

Ito's formula. Let f be a twice continuously differentiable function. Then

$$f(B(t))-f(B(0))=\int_0^t f'(B(s))dB(s)+\frac{1}{2}\int_0^t f''(B(s))ds.$$

Proof. Clearly $f(B(t))-f(B(0))=\sum_{i=0}^{nt-1} f(B(\frac{i+1}{n}))-f(B(\frac{i}{n}))$. Since f is twice differentiable we have by Taylor's theorem,

$$\begin{aligned}f(B(\frac{i+1}{n}))-f(B(\frac{i}{n}))=&f'(B(\frac{i}{n}))(B(\frac{i+1}{n})-B(\frac{i}{n}))\\&+\frac{1}{2}f''(B(\frac{i}{n}))(B(\frac{i+1}{n})-B(\frac{i}{n}))^2\\&+o(B(\frac{i+1}{n})-B(\frac{i}{n}))^2.\end{aligned}$$

By the definition of the stochastic integral

$$\sum_{i=0}^{nt-1} f'(B(\frac{i}{n}))(B(\frac{i+1}{n})-B(\frac{i}{n}))\to\int_0^t f'(B(s))dB(s).$$

For the next term we can write

$$E[\left(\sum_{i=0}^{nt-1} f''(B(\frac{i}{n}))(B(\frac{i+1}{n})-B(\frac{i}{n}))^2-\frac{1}{n}\sum_{i=0}^{nt-1} f''(B(\frac{i}{n}))\right)^2]=\sum_{i,j=0}^{nt-1} E[X_iX_j]$$

where $X_i=f''(B(\frac{i}{n}))((B(\frac{i+1}{n})-B(\frac{i}{n}))^2-\frac{1}{n})$. If $i<j$ then preconditioning on $\mathcal{F}(\frac{j}{n})$ gives

$$E[f''(B(\frac{i}{n}))((B(\frac{i+1}{n})-B(\frac{i}{n}))^2-\frac{1}{n})f''(B(\frac{j}{n}))E[(B(\frac{j+1}{n})-B(\frac{j}{n}))^2-\frac{1}{n}\mid\mathcal{F}(\frac{j}{n})]=0$$

so all the off-diagonal terms just disappear. For the on-diagonal terms $E[(X_i)^2]$ preconditioning on $\mathcal{F}(\frac{i}{n})$ gives

$$E[(f''(B(\frac{i}{n})))^2 E[(((B(\frac{i+1}{n}) - B(\frac{i}{n}))^2 - \frac{1}{n})^2 \mid \mathcal{F}(\frac{i}{n})]].$$

Direct computation gives

$$E[(((B(\frac{i+1}{n}) - B(\frac{i}{n}))^2 - \frac{1}{n})^2 \mid \mathcal{F}(\frac{i}{n})] = cn^{-2} \qquad \text{where } \; c = \int (x^2-1)^2 e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}.$$

The point is that for some constant C,

$$E[\left(\sum_{i=0}^{nt-1} f''(B(\frac{i}{n}))(B(\frac{i+1}{n}) - B(\frac{i}{n}))^2 - \frac{1}{n}\sum_{i=0}^{nt-1} f''(B(\frac{i}{n}))\right)^2] \leq Cn^{-1}.$$

In particular $\sum_{i=0}^{nt-1} f''(B(\frac{i}{n}))(B(\frac{i+1}{n})-B(\frac{i}{n}))^2$ must have the same limit as $\frac{1}{n}\sum_{i=0}^{nt-1} f''(B(\frac{i}{n}))$ which is a normal Riemann sum approximation for $\int_0^t f''(B(s))ds$. So the second term in Taylor's expansion gives

$$\sum_{i=0}^{nt-1} f''(B(\frac{i}{n}))(B(\frac{i+1}{n}) - B(\frac{i}{n}))^2 \to \int_0^t f''(B(s))ds.$$

This is really the same argument as we used the compute the quadratic variation of Brownian motion. The same argument shows that the lower order terms go to zero, and we are left with Ito's formula.

Ito's formula for functions which depend on time as well. Let $f(x,t)$ have two continuous space derivatives and one continuous time derivative. Then

$$f(B(t),t) - f(B(0),0) = \int_0^t \frac{\partial f}{\partial x}(B(s),s)dB(s) + \int_0^t [\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}](B(s),s)ds.$$

The proof is exactly the same as before.

Ito's formula for d-dimensional Brownian motion. d-dimensional Brownian motion $B(t)$ just means a vector $(B_1(t),\ldots,B_d(t))$ of d independent one dimensional Brownian motions. By the vector version of Taylor's theorem we get Ito's formula

$$\begin{aligned} f(B(t),t) - f(B(0),0) &= \int_0^t \sum_{i=1}^d \frac{\partial f}{\partial x_i}(B(s),s)dB_i(s) + \int_0^t [\frac{\partial f}{\partial t} + \frac{1}{2}\sum_{i=1}^d \frac{\partial^2 f}{\partial x_i^2}](B(s),s)ds \\ &= \int_0^t \nabla f \cdot dB(s) + \int_0^t [\frac{\partial f}{\partial t} + \frac{1}{2}\Delta f]ds. \end{aligned}$$

Appendix. We want to prove that if $E[\int_0^T |\delta(u)|^2 du] < \infty$ and $\delta(t)$ is adapted to $\mathcal{F}(t)$ and $\epsilon > 0$ then there exists an adapted process $\gamma(t)$ which is simple in the sense that there are (non-random) times $0 = t_0 \le t_1 \le \cdots \le t_m = T$ and random variables $e_0, e_1, \ldots, e_{m-1}$ such that

$$\gamma(t,\omega) = e_i(\omega) \qquad \text{for } \; t_i \le t < t_{i+1}$$

and for which

$$\|\gamma - \delta\|^2 = E[\int_0^T |\gamma(u) - \delta(u)|^2 du] < \epsilon.$$

Proof. Let $\delta_n(t) = \delta(t)$ if $|\delta(t)| \le n$ and $\delta_n(t) = n$ if $\delta(t) \ge n$, $\delta_n(t) = -n$ if $\delta(t) \le -n$ (i.e. cut off the high bits). Then its easy to check that δ_n is adapted and $\|\delta_n - \delta\| \to 0$ as $n \to \infty$. So choose n_0 big enough that $\|\delta_{n_0} - \delta\| < \epsilon/3$ and call $\delta_{n_0}(t) = \gamma^1(t)$.

Let $\gamma_n^1(t) = n \int_{t-1/n}^t \gamma^1(u)du$. Since $\gamma^1(t)$ is bounded, each $\gamma_n^1(t)$ is continuous and since the integration was over a little time step *backwards* in time all the $\gamma_n^1(t)$ are still adapted to $\mathcal{F}(t)$. Also we have

$$\|\gamma_n^1 - \gamma^1\| \to 0 \qquad \text{as } \; n \to \infty.$$

So we can choose n_1 big enough that $\|\gamma_{n_1}^1 - \gamma^2\| \le \epsilon/3$ and call $\gamma_{n_1}^1(t) = \gamma^2(t)$.

Let $\gamma_n^2(t) = \gamma^2(\frac{i}{n})$ for $\frac{i}{n} \le t < \frac{i+1}{n}$. Then each γ_n^2 is adapted, simple and, since γ^2 is continuous, $\int_0^T |\gamma_n^2(u) - \gamma^2(u)|^2 du \to 0$ as $n \to \infty$ for each ω. Since γ^2 is also bounded we can invoke the bounded convergence theorem to conclude that $\|\gamma_n^2 - \gamma^2\|^2 = E[\int_0^T |\gamma_n^2(u) - \gamma^2(u)|^2 du] \to 0$ as well. Choosing n_2 so large that $\|\gamma_{n_2}^2 - \gamma^2\| < \epsilon/3$ and letting $\gamma(t) = \gamma_{n_2}^2(t)$ we finally have

$$\|\gamma - \delta\| \le \|\gamma - \gamma^2\| + \|\gamma^2 - \gamma^1\| + \|\gamma^1 - \delta\| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

References. Chapters 14 and 15 of Shreve's notes; 3.1,3.2,4.1 of Oksendal; 3.2A,B, 3.3A of Karatzas & Shreve.

Problems.

1. Suppose that $\delta(t)$ is a non-random continuously differentiable function and $B(t)$ is a Brownian motion. Show that $\int_0^t \delta(u)dB(u) = \delta(t)B(t) - \int_0^t \delta'(u)B(u)du$.
2. Let $S(t) = S(0)e^{\{\sigma B(t) + (\mu - \frac{1}{2}\sigma^2)t\}}$ be a geometric Brownian motion (stock price process). Let $\mathcal{G}(t)$ be the σ-field generated by $S(u)$, $0 \le u \le t$.
   i. In what way does the filtration $\mathcal{G}(t)$ differ from that of the original Brownian motion $B(t)$?
   ii. Let $\delta(t)$ be the number of shares held at time t. Explain how to construct the stochastic integral representing your wealth $\int_0^t \delta(u)dS(u)$ and what are reasonable assumptions to make on δ.
   iii. Compute the instantaneous absolute volatility of the wealth process, and find the analogue of the L^2 isometry which holds for geometric Brownian motion.
   iv. Compute the instantaneous absolute volatility of the logarithm of the wealth process. Which definition of volatility is the correct one from the point of view of finance?

3. In the Black-Scholes analysis $\Delta(t) = \frac{\partial V}{\partial S}(S(t),t)$ is the hedge ratio. Let k be the *relative roundtrip transaction cost* (a constant). Then the *total transaction cost of the rehedging strategy* is

$$\int_0^t \frac{k}{2} S(u)|d\Delta(u)|.$$

Clearly this should be defined as the limit of

$$\frac{k}{2}\sum_{i=0}^{nt-1} S(\frac{i}{n})|\Delta(\frac{i+1}{n}) - \Delta(\frac{i}{n})|$$

as $n \to \infty$.

i. Compute the limit, where $S(t)$ is a geometric Brownian motion and explain what the answer means in financial terms.

ii. Show that the leading order asymptotics is governed by the second derivative of the value process with respect to price. Under what conditions is the limit finite?