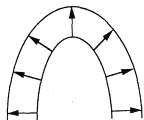


The Kardar-Parisi-Zhang Equation and Universality Class

October 13, 2012

Kardar-Parisi-Zhang Equation (1986)

$$\partial_t h = - \underbrace{\frac{1}{2}(\partial_x h)^2}_{\text{lateral growth}} + \underbrace{\frac{1}{2}\partial_x^2 h}_{\text{relaxation}} + \underbrace{\xi}_{\text{space-time white noise}} \underbrace{h(t, x)}_{\substack{\text{height at time} \\ \text{t position x}}}$$



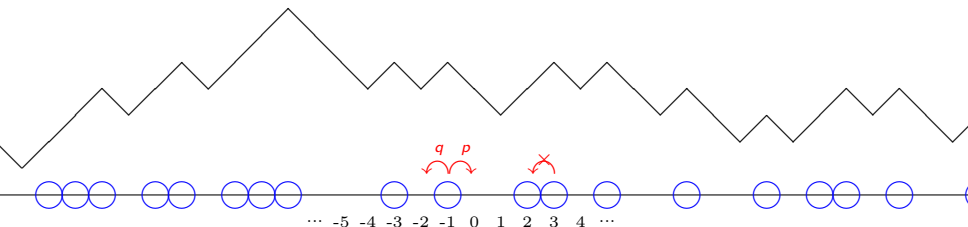
$$F(\partial_x h) \sim F(0) + F'(0)\partial_x h + \frac{1}{2}F''(0)(\partial_x h)^2 + \dots$$

- ▶ model eqn for stochastic interface growth with local evolution
- ▶ nothing to do with other KPZ (Knizhnik-Polyakov-Zamolodchikov) (so far...)
- ▶ we only consider $d = 1$ here (1 + 1 dimensions)
- ▶ Brownian motion invariant, modulo height shift; all solutions locally Brownian
- ▶ nonstandard fluctuations of size $t^{1/3}$ (Forster-Nelson-Stephen 77) for equivalent stochastic Burgers $\partial_t u = -u\partial_x u + \frac{1}{2}\partial_x^2 u + \partial_x \xi$, $u = \partial_x h$
- ▶ models a wide variety of systems (KPZ universality class)
- ▶ Exact formulas (!)

Height fn of ASEP (Asymmetric Simple Exclusion Process)

local min $\mapsto$ local max at rate q

local max $\mapsto$ local min at rate p

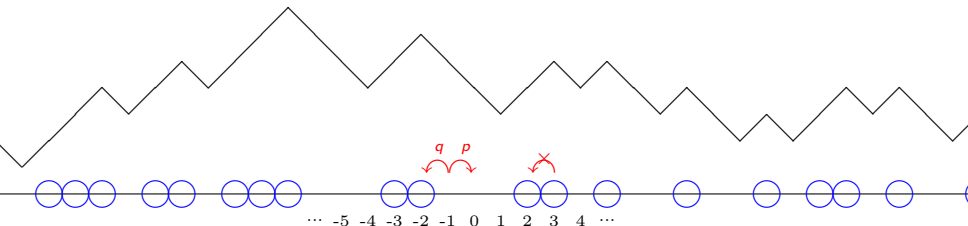


symmetric random walk invariant (except for height shift)

Height fn of ASEP (Asymmetric Simple Exclusion Process)

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symmetric random walk invariant (except for height shift)

Directed random polymers in $1 + 1$ dimensions

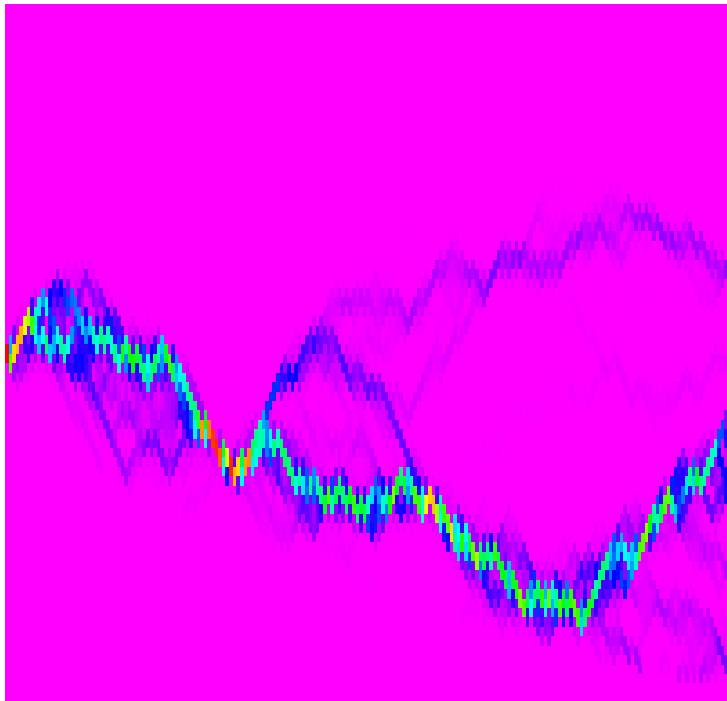
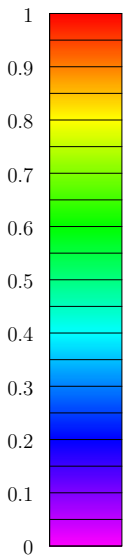
Point-to-point polymer is probability measure

$$P_N^\xi \sim \frac{e^{-\beta \sum_{i=1}^N \xi(i, x(i))}}{Z_\beta^\xi(N, x)}$$

on nearest neighbour random walk paths $x(\cdot)$ with $x(0) = 0$, $x(N) = x$
 $\xi(i, j)$ independent, identically distributed random variables

Free energy $\log Z_\beta^\xi(N, x)$ satisfies discrete KPZ – like equation

Invariant measure unknown except in special case where ξ have
log-Gamma distribution (Seppalainen 10)



Making sense of the KPZ eqn $\partial_t h = -\frac{1}{2}(\partial_x h)^2 + \frac{1}{2}\partial_x^2 h + \xi$

$h(t, x)$ locally Brownian $\Rightarrow (\partial_x h)^2 = \infty$ so cannot make sense

Attempts to replace nonlinearity by Wick ordered version led to nonphysical solutions

Bertini-Giacomin 97 Define $h(t, x) = -\log Z(t, x)$ where

$$\partial_t Z = \frac{1}{2}\partial_x^2 Z - \xi Z \quad (\text{well-posed Ito equation})$$

Gives

$$\partial_t h = -\frac{1}{2}[(\partial_x h)^2 - \infty] + \frac{1}{2}\partial_x^2 h + \xi \quad (*)$$

in sense that $Z > 0$, Z is the limit of Z_ϵ solving

$$\partial_t Z_\epsilon = \frac{1}{2}\partial_x^2 Z_\epsilon - \underbrace{\xi_\epsilon}_{\text{mollified in space}}$$

and $h_\epsilon = -\log Z_\epsilon$ solves

$$\partial_t h_\epsilon = -\frac{1}{2}[(\partial_x h_\epsilon)^2 - \underbrace{C_\epsilon}_{\text{divergent}}] + \frac{1}{2}\partial_x^2 h_\epsilon + \xi_\epsilon$$

Thm (Hairer 11) $(*)$ is well-posed

Rescaling the KPZ eqn $h_\epsilon(t, x) := \epsilon^b h(\epsilon^{-z}t, \epsilon^{-1}x)$

satisfies $\partial_t h_\epsilon = -\frac{1}{2}\epsilon^{2-z-b}(\partial_x h_\epsilon)^2 + \frac{1}{2}\epsilon^{2-z}\partial_x^2 h_\epsilon + \epsilon^{b-\frac{z}{2}+\frac{1}{2}}\xi$

KPZ scaling $b = 1/2, z = 3/2$ gives

$$\partial_t h_\epsilon = -\frac{1}{2}(\partial_x h_\epsilon)^2 + \frac{1}{2}\epsilon^{1/2}\partial_x^2 h_\epsilon + \epsilon^{1/4}\xi$$

Formally get Burgers eqn $\partial_t h_0 = -\frac{1}{2}(\partial_x h_0)^2$ in limit, but entropy solns do *not* preserve Brownian motion. As we will see, noise survives in limit

Weakly asymmetric scaling $b = 1/2, z = 2$, *but start with nonlinearity of $\mathcal{O}(\epsilon^{1/2})$* . Gives back standard KPZ

Intermediate disorder scaling $b = 0, z = 2$, *but start with noise of $\mathcal{O}(\epsilon^{1/2})$* . Gives back standard KPZ

Last two scalings useful to obtain KPZ eqn from models with adjustable parameter (eg. weakly asymmetric exclusion $q - p = \epsilon^{1/2}$, intermediate disorder directed polymers $\beta = O(\epsilon^{1/2})$)

Weakly asymmetric and intermediate disorder limits

Thm (Bertini-Giacomin 97, Amir-Corwin-Q 10) Start ASEP with $h^{\text{ASEP}}(0, x) = |x|$, ie. sites to the right of 0 occupied. Then

$$\epsilon^{1/2} h_{q-p=\epsilon^{1/2}}^{\text{ASEP}}(\epsilon^{-2}t, \epsilon^{-1}x) - \frac{1}{2}\epsilon^{-1}t - \log(\frac{1}{2}\epsilon^{-1/2}) \rightarrow h(t, x)$$

where $h(t, x)$ is the *narrow wedge solution of KPZ*, i.e.

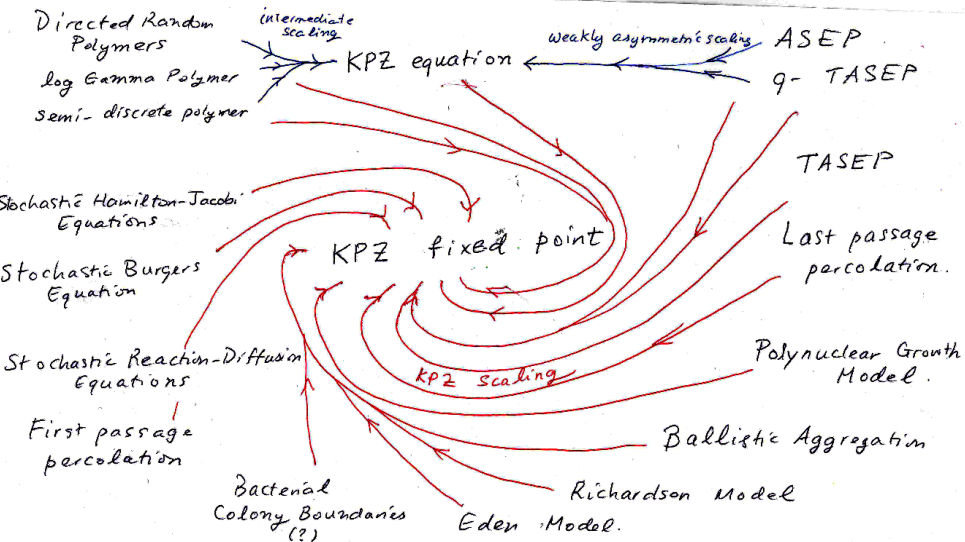
$h(t, x) = -\log Z(t, x)$ where $Z(t, x)$ is the solution of the stochastic heat eqn $\partial_t Z = \frac{1}{2}\partial_x^2 Z - \xi Z$ with $Z(0, x) = \delta_0(x)$.

Thm (Alberts-Khanin-Q 11) Let $Z_\beta(N, x) = E_0[e^{-\beta \sum_{i=1}^N \xi(i, x(i))} \mathbf{1}_{x(N)=x}]$ expectation over nearest neighbour random walk paths starting at 0. Assume $E[e^{s\xi(i,j)}] = e^{\lambda(s)} < \infty$ ($E[\xi^6(i,j)] < \infty$ should suffice). Then

$$-\log Z_{(\epsilon/2)^{1/2}}(\lfloor \epsilon^{-2}t \rfloor, \lfloor \epsilon^{-1}x \rfloor) - \epsilon^{-2}\lambda((\epsilon/2)^{1/2})t - \log(\frac{1}{2}\epsilon^{-1}) \rightarrow h(t, x)$$

Can get general initial data for KPZ by adjusting initial conditions for ASEP or boundary conditions on the polymer

KPZ Universality Class



Weakly asymmetric/intermediate disorder scaling $\rightsquigarrow$ KPZ eqn (well understood)

KPZ scaling $\epsilon^{1/2} h(\epsilon^{-3/2} t, \epsilon^{-1} x) \rightsquigarrow$ KPZ fixed point (nontrivial, poorly understood)

Exact distribution functions breakthrough 99

$\beta \rightarrow \infty$ limit of directed polymer $\mapsto$ Last Passage Percolation

$$G(M, N) = \max_{\substack{\text{up/right paths} \\ (1,1) \text{ to } (M,N)}} \sum_{(i,j) \in \text{path}} \xi(i, j),$$

free energy satisfies discrete KPZ

$$G(M, N) = \max(G(M-1, N), G(M, N-1)) + \xi(M, N)$$

If $\xi(i, j)$ have geometric distr $P(G(M, N) \leq s) = \det(I - K_N)_{L^2(s, \infty)}$
and asymptotics gives (Johansson 00; see also Baik-Deift-Johansson 99)

$$G(N, N) \sim c_1 N + c_2 N^{1/3} \zeta \quad P(\zeta \leq s) = F_{\text{GUE}}(s) = \det(I - K)_{L^2(s, \infty)}$$

$$K(x, y) = \int_0^\infty \text{Ai}(x+s) \text{Ai}(y+s) ds$$

Same as for largest eigenvalue of GUE matrix

$$a_{ij} = \overline{a_{ji}} \sim \begin{cases} \mathcal{N}(0, \sqrt{N}/2) + i\mathcal{N}(0, \sqrt{N}/2) & i < j = 1, \dots, N \\ \mathcal{N}(0, \sqrt{N}) & i = j \end{cases}$$

00 - 05 special models Geometric LPP/PNG Exponential LPP/TASEP
give broad picture of fluctuations. Models come from determinantal
processes, so appearance of Fredholm determinants not so surprising

Broad picture of fluctuations in KPZ universality class

Point-to-point random growth on curved substrate, stochastic heat eq starting from delta function $Z(0, x) = \delta_0(x)$, KPZ starting from narrow wedge, etc

$$h(t, x) \sim c_1 t + c_2 \frac{x^2}{t} - c_3 t^{1/3} \text{Ai}_2(c_4 t^{-2/3} x)$$

Airy₂ process
stationary, marginals F_{GUE}

$t \rightarrow \infty$ c_i non-universal, in some cases computable

Point-to-line e.g. random growth on flat substrate, stochastic heat eq starting from a constant $Z(0, x) \equiv 1$, KPZ starting from 0, etc

$$h(t, x) \sim c_5 t - c_6 t^{1/3} \text{Ai}_1(c_7 t^{-2/3} x)$$

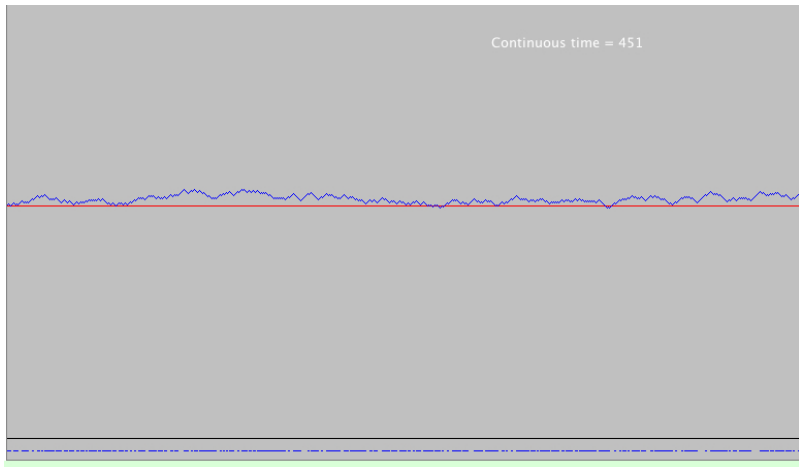
Airy₁ process
stationary, marginals F_{GOE}

Equilibrium case e.g. stochastic Burgers equation with invariant white noise, KPZ with Brownian motion $\text{Ai}_{\text{stat}}(x)$

Mixed cases start with one of the above to the right of the origin, and another to the left. $\text{Ai}_{1 \rightarrow 2}(x)$, $\text{Ai}_{1 \rightarrow \text{stat}}(x)$, $\text{Ai}_{2 \rightarrow \text{stat}}(x)$

Six KPZ universality subclasses

Limiting processes $\text{Ai}_\bullet(x)$ all have determinantal finite dimensional distributions



Approximate Airy_1 process. At each x difference from red line has approximate F_{GOE} distribution

Experimental evidence! Takeuchi, Sano 10

Was difficult to find repeatable, local dynamics with approximate space-time white noise. This one uses nematic liquid crystals

2

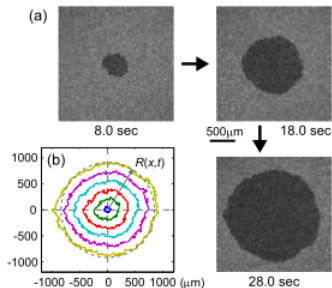
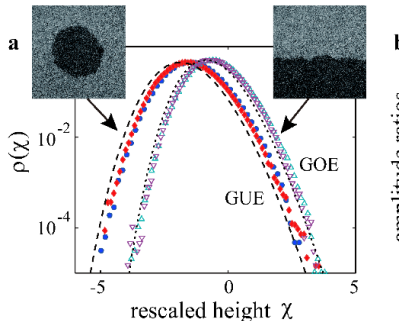


FIG. 1: (Color online) Growing DSM2 cluster. (a) Images. Indicated below is the elapsed time after the emission of laser pulses. (b) Snapshots of the interfaces taken every 5 s in the range $2 \text{ s} \leq t \leq 27 \text{ s}$. The gray dashed circle shows the mean radius of all the droplets at $t = 27 \text{ s}$. The coordinate x at this time is defined along this circle.



Not only confirms $t^{1/3}$ size of fluctuations, but statistically distinguishes F_{GUE} from F_{GOE} by skewness and kurtosis
New experiments of Yunker et al find it in drying coffee rings!

Fluctuations of the KPZ equation

So far, only determinantal models LPP/TASEP

What about KPZ equation itself?

By refining the coupling method (Cator-Groeneboom 06, Balazs-Seppalainen 06) we can take weakly asymmetric limit of exclusion to get

Thm (Balazs-Q-Seppalainen 09) Let $h(t, x)$ be the solution of the KPZ equation starting from a two-sided Brownian motion. Then there are $0 < C_1$ and $C_2 < \infty$ such that

$$C_1 t^{1/3} \leq \sqrt{\text{Var}(h(t, 0))} \leq C_2 t^{1/3}$$

How does one get the fluctuation exponent?

In any dimension $P(x(t) \in dx) = \frac{Z(t,x)dx}{\int Z(t,y)dy} \sim Ce^{-\frac{|x|^2}{2t} + t\chi_{\text{fluc}}\zeta}$ so

$$E[|x(t)|] := Ct^{\chi_{\text{wander}}}$$

gives the KPZ relation: $2\chi_{\text{wander}} - 1 = \chi_{\text{fluc}}$

In $d = 1$ we have the extra relation (inherited from ASEP)

$$\text{Var}(h(t, 0)) = E[|x(t)|]$$

ie.

$$t^{2\chi_{\text{fluc}}} \sim t^{\chi_{\text{wander}}}$$

ie.

$$2\chi_{\text{fluc}} = \chi_{\text{wander}}$$

which forces $\chi_{\text{fluc}} = 1/3$, $\chi_{\text{wander}} = 2/3$.

Real proof uses ordering of particles to estimate probability of second class particle in wasp $x(t) > C\epsilon^{1/3}t^{2/3}$ in terms of current and then uses Chebyshev and $\text{Var}(\text{Curr}) = E[|x(t)|]$ to get a closed inequality for moments of $x(t)$ (up to 3)

Identifies correct order, but not likely to get distribution of fluctuations

Breakthrough was Tracy-Widom formulas for ASEP 08

e.g. Start with particles at $\mathbb{Z}_+$, ie. $h^{ASEP}(0, x) = |x|$

$$P\left(h^{ASEP}\left(\frac{t}{q-p}, x\right) \geq y\right) = \frac{1}{2\pi i} \oint \frac{d\mu}{\mu} \prod_{k=0}^{\infty} (1 - \mu\left(\frac{p}{q}\right)^k) \det(I - J)$$

$$J(\eta, \eta') = \oint e^{\Psi(\zeta) - \Psi(\eta')} \sum_{k=-\infty}^{\infty} \frac{\left(\frac{p}{q}\right)^k \left(\frac{\zeta}{\eta'}\right)^k}{\mu - \left(\frac{p}{q}\right)^k} \frac{d\zeta}{\eta'(\zeta - \eta)}$$

$$\Psi(\zeta) = -x \log(1 - \zeta) + \frac{t\zeta}{(1-\zeta)} + (y + x) \log \zeta$$

- ▶ appearance of Fredholm determinant unexpected
- ▶ similar formula for step Bernoulli initial data
- ▶ miracle of Fredholm determinant not found for other models, so no hope (yet) of doing asymptotics
- ▶ more direct proofs now available using duality ([Borodin-Corwin-Sasamoto 12](#))

Exact formulas for one-point distribution of KPZ at finite t

$$\epsilon^{1/2} h_{q-p=\epsilon^{1/2}}^{\text{ASEP}}(\epsilon^{-2}t, \epsilon^{-1}x) - \frac{1}{2}\epsilon^{-1}t - \log(\frac{1}{2}\epsilon^{-1/2}) \rightarrow h(t, x)$$

limit in Tracy-Widom formula by steepest descent gives

Thm (independently by Amir-Corwin-Q (rigorous) & Sasamoto-Spohn (non-rigorous) 10) Exact formula for $P(h(t, x) \leq s)$ where $h(t, x)$ is the solution of KPZ with narrow wedge initial data. Generating fn is slightly easier to digest

$$E \left[\exp \left\{ -e^{-s} e^{-h(t, x) - \frac{x^2}{2t} + \frac{t}{24}} \right\} \right] = \det(I - P_0 K_{s, t} P_0)_{L^2(\mathbb{R})}$$

$$K_{s, t}(x, y) = \int_{-\infty}^{\infty} \frac{1}{1 + e^{-(t/2)^{1/3} \lambda + s}} \text{Ai}(x + \lambda) \text{Ai}(y + \lambda) d\lambda$$

As $t \rightarrow \infty$, $-h(t, x) - \frac{x^2}{2t} + \frac{t}{24} \sim 2^{-1/3} t^{1/3} \zeta$ where ζ has the F_{GUE} distribution. As $t \rightarrow 0$, it is $t^{1/4} \times \text{Gaussian}$

Thm (Corwin-Q 10) Analogous formula for half-Brownian initial data, i.e.
 $Z(0, x) = e^{B(x)} \mathbf{1}_{x > 0}$.

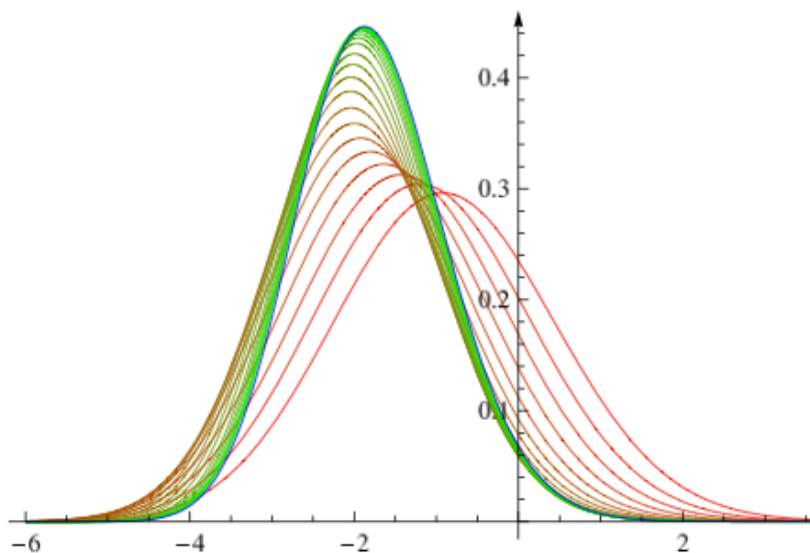


FIGURE 1. Properly scaled probability density functions for $F_T(s)$ show crossover to F_{GUE} in long time (green, upper curves) and Gaussian in short time (red, lower curves). These plots have been made by evaluating the Fredholm determinant for $F_T(s)$ (the function $\det(1 - T_s)$) for various values of T .

Cor (Weak universality for directed random polymers) (Alberts-Khanin-Q 11)

From the exact formula for KPZ and the intermediate disorder limit, we have, for an explicit $c(\beta, N)$,

$$\lim_{\beta \rightarrow \infty} \lim_{N \rightarrow \infty} \beta^{-4/3} \left[\log E_x \left[e^{-\frac{\beta}{N^{1/4}} \sum_{i=0}^N \xi(i, x(i))} \right] - c(\beta, N) \right] \stackrel{\text{distr}}{=} F_{GUE}$$

Key point. Does not depend on distribution of the i.i.d.'s $\xi(i, x)$ (just need a few moments). Compare with the full universality conjecture

$$\lim_{N \rightarrow \infty} N^{-1/3} \left[\log E_x \left[e^{-\beta \sum_{i=0}^N \xi(i, x(i))} \right] - c(\beta)N \right] \stackrel{\text{distr}}{=} F_{GUE}$$

now known for several partially solvable models

- ▶ Continuum random polymer (Amir-Corwin-Q 10, Alberts-Khanin-Q 11)
- ▶ Semi-continuous polymer (discovered by O'Connell-Yor, exact formula for free energy O'Connell 09, asymptotics Borodin-Corwin 11)
- ▶ log-Gamma polymer (discovered by Seppalainen, exact formula for free energy Corwin-O'Connell-Seppalainen-Zygouras 12, asymptotics Borodin-Corwin-Remenik 12)
- ▶ Common algebraic framework yielding Fred det's and new solvable models, eg. q-TASEP (Borodin-Corwin 11)

Broad picture of fluctuations in KPZ universality class

Point-to-point random growth on curved substrate, stochastic heat eq starting from delta function $Z(0, x) = \delta_0(x)$, KPZ starting from narrow wedge, etc

$$h(t, x) \sim c_1 t + c_2 \frac{x^2}{t} - c_3 t^{1/3} \text{Ai}_2(c_4 t^{-2/3} x)$$

Airy₂ process
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Equilibrium case e.g. stochastic Burgers equation with invariant white noise, KPZ with Brownian motion $\text{Ai}_{\text{stat}}(x)$

Mixed cases start with one of the above to the right of the origin, and another to the left. $\text{Ai}_{1 \rightarrow 2}(x)$, $\text{Ai}_{1 \rightarrow \text{stat}}(x)$, $\text{Ai}_{2 \rightarrow \text{stat}}(x)$

Known for TASEP/PNG. For KPZ we have exact formula for one pt distr of $h(t, x)$ for narrow wedge and half-Brownian data. Pt-to-pt log-Gamma and semi-continuous polymer free energy.

Replica method (non-rigorous)

Based on (rigorous) fact that $u(t, x_1, \dots, x_n) = E[Z(t, x_1) \cdots Z(t, x_n)]$ satisfies $\partial_t u = \mathcal{L}u$ where

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^n \partial_{x_i}^2 + \frac{1}{2} \sum_{i \neq j=1}^n \delta(x_i - x_j) \quad \text{attractive } \delta\text{-Bose gas}$$

- ▶ Explicit (rigorous) eigenfunctions/eigenvalues [Bethe ansatz](#)
- ▶ Try to reconstruct distribution of Z from this
 $E[Z^n] \sim e^{n^3}$ and DO NOT determine the distribution
- ▶ KPZ formula quickly rederived this way [\(Dotsenko, Calabrese-LeDoussal-Rosso 10\)](#)
- ▶ n dimensional distributions of KPZ with narrow wedge initial data converge to $\text{Ai}_2(x)$ assuming **factorization approximation on Bethe eigenfunctions** [\(Prolhac-Spohn 11\)](#)
- ▶ formula for KPZ starting from flat [\(Calabrese-LeDoussal 12\)](#)
- ▶ formula for KPZ starting from Brownian motion [\(Imamura-Sasamoto 12\)](#)
- ▶ Recent results [\(Borodin-Corwin-Sasamoto 12\)](#) can be considered a rigorous version of replicas for ASEP, q-TASEP

Variational problems for Airy₂ process

In a sense the Airy₂ process is the 'fundamental solution', and other universal quantities occur as variational problems

e.g. general initial data h_0 for KPZ. By linearity of stochastic heat eqn

$$h(t, x) \sim -\log \int e^{-\frac{(x-y)^2}{2t} + t^{1/3} \text{Ai}_2(t^{-2/3}(x-y)) - h^0(y)} dy$$

Limiting fluctuations all at scale $h_\epsilon(t, x) = \epsilon^{1/2} h(\epsilon^{-3/2} t, \epsilon^{-1} x)$

$$h_\epsilon(t, x) \sim -\epsilon^{1/2} \log \int e^{\epsilon^{-1/2} \left\{ -\frac{(x-y)^2}{2t} + t^{1/3} \text{Ai}_2(t^{-2/3}(x-y)) - \epsilon^{1/2} h^0(\epsilon^{-1} y) \right\}} dy$$

Assume $h_\epsilon(t, x) \rightarrow \mathfrak{h}(t, x)$ (KPZ fixed point)

$$\mathfrak{h}(t, x) = -\sup_y \left\{ -\frac{(x-y)^2}{2t} + t^{1/3} \text{Ai}_2(t^{-2/3}(x-y)) - \mathfrak{h}^0(y) \right\}$$

True in the sense of one-pt distributions, and several cases have now been checked Corwin-Q-Remenik 11, Q-Remenik 11

Conjectural KPZ fixed point formula

To see multi(space)point distributions need an extra parameter.

Let $Z(t, x; y)$ be the solution at (t, x) of the stochastic heat eqn with initial data $\delta_y(x)$. Then

$$Z(t, x; y) \sim c_t e^{-\frac{(x-y)^2}{2t}} + t^{1/3} \mathcal{A}(t^{-2/3}x, t^{-2/3}y)$$

$\mathcal{A}(x, y)$ is the **Airy sheet**. By time-reversal symmetry, it is an Ai_2 process in each coordinate. By semi-group property, if $\tilde{\mathcal{A}}$ is independent copy

$$2^{1/3} \mathcal{A}(2^{-2/3}x, 2^{-2/3}y) - \frac{(x-y)^2}{4} \stackrel{\text{distr}}{=} \sup_{z \in \mathbb{R}} \left\{ \mathcal{A}(x, z) - \frac{(x-z)^2}{2} + \tilde{\mathcal{A}}(z, y) - \frac{(z-y)^2}{2} \right\}$$

Thm (Cator-Corwin-Pimentel-Q, in prep) In Poissonian LPP such a limiting object exists. **Uniqueness/universality is the problem.**

For fixed $t > 0$ the KPZ fixed point formula reads

$$\mathfrak{h}(t, x) = - \sup_y \left\{ -\frac{(x-y)^2}{2t} + t^{1/3} \mathcal{A}(t^{-2/3}x, t^{-2/3}y) - \mathfrak{h}^0(y) \right\}$$

Lax-Oleinik formula for Burgers eq driven by Airy noise

To get multi(time) distributions need to add time parameters to the Airy sheet $\mathcal{A}(s, x, t, y)$

Proposed formula for transition probs of KPZ fixed point

If one believes **Prolhac-Spohn factorization approximation** then the full transition probabilities can be computed by replicas (Corwin-Q 11)

$$P(h(t, x) \leq g(x) \text{ on } [a, b] \mid h(0, x) = f(x) \text{ on } [c, d]) = \det(I - K + L)$$

where $L(z, z')$ is given by

$$\int \int dm du e^{(d-c)H} K(\partial_m Y_{[c,d]}^{\tilde{f}(\cdot)+m})(z', u) Y_{[a,b]}^{\tilde{g}(\cdot)+u-m} e^{(b-a)H} K(u, z)$$

$Y_{[a,b]}^f$ is solution operator of
$$\begin{cases} \partial_t u = -Hu & a < t < b \\ u(t, x) = 0 & x \geq f(t) \end{cases}$$

$H = -\partial_x^2 + x$ is the Airy operator

Hard to verify.

Test: Flat $\mapsto$ multipoint should give Ai_1 . One pt ok (Corwin-Q-Remenik 11)
but 2 point function unclear. Passes some non-trivial tests, but even numerics is inconclusive (Corwin-Q-Remenik in prep)

Polymer endpoint distribution

Recall the conjecture: Start stochastic heat eq $\partial_t Z = \frac{1}{2} \partial_x^2 Z - Z \xi$ with $\delta_0(x)$

$$Z(t, x) \sim c_t e^{-\frac{x^2}{2t} + 2^{-1/3} t^{1/3} \text{Ai}_2(2^{-1/3} t^{-2/3} x)}$$

so

$$P_t^\xi(x(t) = x) = \frac{Z(t, x)}{\int Z(t, x) dx} \sim \frac{e^{-\frac{x^2}{2t} + 2^{-1/3} t^{1/3} \text{Ai}_2(2^{-1/3} t^{-2/3} x)}}{\text{normalization}}$$

If we change variables

$$\tilde{x} = 2^{-1/3} t^{-2/3} x$$
$$\frac{Z(t, x)}{\int Z(t, x) dx} \sim \frac{e^{2^{-1/3} t^{1/3} \{\text{Ai}_2(\tilde{x}) - \tilde{x}^2\}}}{\text{normalization}} \sim \delta_{\text{argmax}_x \{\text{Ai}_2(x) - x^2\}}$$

Thus $\text{argmax}_x \{\text{Ai}_2(x) - x^2\}$ is the universal polymer endpoint distribution in $1 + 1$ dimensions. Johansson 02: Is there a formula for its law?

The Airy₂ process

- ▶ scaling limit of the top eigenvalue in matrix valued diffusion for GUE (Dyson's Brownian motion)
- ▶ derived via transfer matrix method for PNG, TASEP

(Prahofer-Spohn 02)

- ▶ defined through n -dimensional distributions

$$P(\text{Ai}_2(t_1) \leq \xi_1, \dots, \text{Ai}_2(t_m) \leq \xi_m)$$

given by Fredholm determinant on $L^2(\mathbb{R}^n)$ of *extended kernel*

$$f(t_j, x) = \mathbf{1}_{x \in (\xi_j, \infty)}$$

$$P(\text{Ai}_2(t_1) \leq \xi_1, \dots, \text{Ai}_2(t_m) \leq \xi_m) = \det(I - f^{1/2} K_{\text{ext}} f^{1/2})_{L^2(\{t_1, \dots, t_m\} \times \mathbb{R})}$$

$$K_{\text{ext}}(t, \xi; t', \xi') = \begin{cases} \int_0^\infty d\lambda e^{-\lambda(t-t')} \text{Ai}(\xi + \lambda) \text{Ai}(\xi' + \lambda) & t \geq t' \\ \int_{-\infty}^0 d\lambda e^{-\lambda(t-t')} \text{Ai}(\xi + \lambda) \text{Ai}(\xi' + \lambda) & t < t' \end{cases}$$

- ▶ formulas for the other Airy processes are similar

The Airy₂ process

- ▶ scaling limit of the top eigenvalue in matrix valued diffusion for GUE (Dyson's Brownian motion)
- ▶ derived via transfer matrix method for PNG, TASEP

(Prahofer-Spohn 02)

- ▶ defined through n -dimensional distributions

$$P(\text{Ai}_2(t_1) \leq \xi_1, \dots, \text{Ai}_2(t_m) \leq \xi_m)$$

given by Fredholm determinant on $L^2(\mathbb{R}^n)$ of *extended kernel*

$$P(\text{Ai}_2(t_1) \leq \xi_1, \dots, \text{Ai}_2(t_m) \leq \xi_m) = \det(I - f^{1/2} K_{\text{ext}} f^{1/2})_{L^2(\{t_1, \dots, t_m\} \times \mathbb{R})}$$

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- ▶ formulas for the other Airy processes are similar

Complexity depends on m ! How are we going to compute distribution of $\arg\max_x \{\text{Ai}_2(x) - x^2\}$?

start with alternate formula (Prahofer Spohn) for $P(\text{Ai}_2(t_0) \leq x_0, \dots, \text{Ai}_2(t_n) \leq x_n)$

$$\det(I - K + \bar{P}_{x_0} e^{(t_0-t_1)H} \bar{P}_{x_1} e^{(t_1-t_2)H} \dots \bar{P}_{x_n} e^{(t_n-t_0)H} K)$$

$H = -\partial_x^2 + x$ is the Airy operator $\bar{P}_a f(x) = 1_{(-\infty, a]}(x) f(x)$

Take $t_0 < t_1 < \dots < t_n$ to be a fine mesh in $[\ell, r]$ to get continuum formula

$$P(\text{Ai}_2(t) \leq g(t) \text{ for } t \in [\ell, r]) = \det(I - K + \Theta_{[\ell, r]}^g e^{(r-\ell)H} K)$$

where $\Theta_{[\ell, r]}^g f(x) = u(r, x)$ is solution operator at time r of boundary value problem

$$\left\{ \begin{array}{ll} \partial_t u + Hu = 0 & x < g(t) \\ u(t, x) = 0 & x \geq g(t) \end{array} \right\} \quad \text{with } u(\ell, x) = f(x)$$

Parabolic shift transforms it into standard heat equation

$$\Theta_{[\ell, r]}^g(x, y) = e^{\ell x - ry + (r^3 - \ell^3)/3} \frac{e^{-(x-y)^2/4(r-\ell)}}{\sqrt{4\pi(r-\ell)}} \\ \cdot P_{B(\ell)=x-\ell^2, B(r)=y-r^2}(B(s) \leq g(s) - s^2 \text{ on } [\ell, r]),$$

Perhaps you imagine it will not be so much fun to compute

$$P(\text{Ai}_2(t) \leq g(t) \text{ for } t \in [\ell, r]) = \det(I - K + \Theta_{[\ell, r]}^g e^{(r-\ell)H} K)$$

$$\Theta_{[\ell, r]}^g(x, y) = \frac{e^{\ell x - ry + \frac{r^3 - \ell^3}{3} - \frac{(x-y)^2}{4(r-\ell)}}}{\sqrt{4\pi(r-\ell)}} P_{\substack{x-\ell^2 \\ y-r^2}}(B(s) \leq g(s) - s^2 \text{ on } [\ell, r])$$

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But suppose we are interested in $g(t) = t^2 + m$. By reflection principle, solution operator is completely explicit:

$$\Theta_{[-L, L]}^{t^2+m} = \bar{P}_{m+L^2} e^{-2LH} \bar{P}_{m+L^2} - \bar{P}_{m+L^2} R_L \bar{P}_{m+L^2}$$

where R_L is the reflection term

$$R_L(x, y) = \frac{1}{\sqrt{8\pi L}} e^{-(x+y-2m-2L^2)^2/8L - (x+y)L + 2L^3/3}$$

Amazingly, for any $L > 0$,

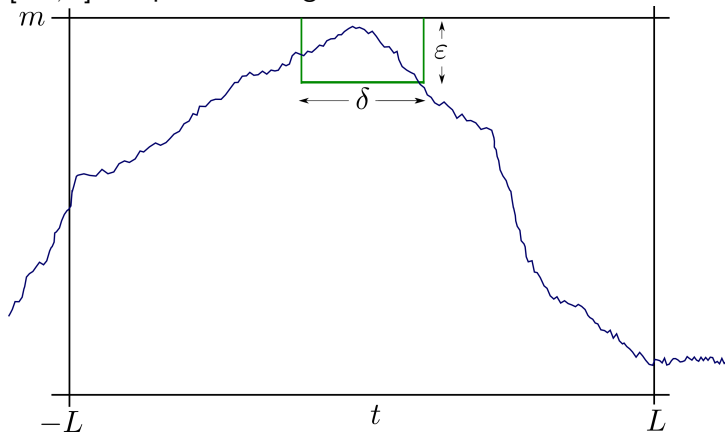
$$\det(I - R_L e^{2LH} K) = F_{\text{GOE}}(4^{1/3} m)$$

Rest vanishes as $L \rightarrow \infty$ by steepest descent. This reproves Johansson's famous result that $\max_y \{\text{Ai}_2(y) - y^2\} \sim F_{\text{GOE}}$ (Corwin-Q-Remenik 11)

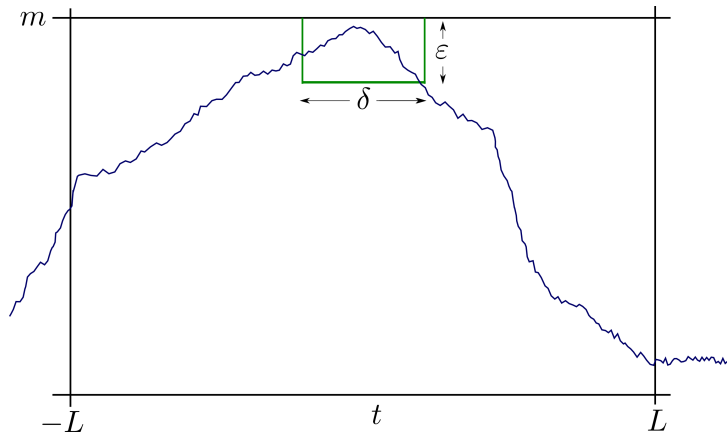
Next we compute joint density of $\operatorname{argmax}_y \{ \text{Ai}_2(y) - y^2 \}$ and $\max_y \{ \text{Ai}_2(y) - y^2 \}$ ([Moreno-Flores, Q, Remenik \(2011\)](#))

$$f(t, m) = \lim_{L \rightarrow \infty} \lim_{\delta \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon \delta} P(A_{t, m, \epsilon, \delta, L})$$

where $A_{t, m, \epsilon, \delta, L}$ is the event that $\text{Ai}_2(x) - x^2$ stays below m on $[-L, L]$ but passes through the window



$$P(A_{t,m,\epsilon,\delta,L}) = P(\text{Ai}_2(x) \leq x^2 + m, |x| \leq L) \\ - P(\text{Ai}_2(x) \leq x^2 + m - \epsilon 1_{|x-t| \leq \delta/2}, |x| \leq L)$$



2nd operator factorizes $\Theta_{[t+\frac{\delta}{2}, L]}^{t^2+m} \Theta_{[t-\frac{\delta}{2}, t+\frac{\delta}{2}]}^{t^2+m-\epsilon} \Theta_{[-L, t-\frac{\delta}{2}]}^{t^2+m}$ and middle becomes rank 1 in limit

Thm [\(MorenoFlores-Q-Remenik June11\)](#) jt density of $\operatorname{argmax}_y\{\operatorname{Ai}_2(y) - \frac{y^2}{2}\}$
and $\max_y\{\operatorname{Ai}_2(y) - \frac{y^2}{2}\}$ is

$$f(t,m)=\gamma(t,4^{1/3}m)F_{\operatorname{GOE}}(4^{1/3}m)$$

$$\gamma(t,m)=2^{1/3}\langle\psi_{-t,4^{-1/3}m},(I-P_0B_mP_0)^{-1}\psi_{t,4^{-1/3}m}\rangle_{L^2(0,\infty)}$$

$$\psi_{t,m}(x)=2e^{2^{1/3}xt}\left[t\operatorname{Ai}(2^{1/3}x+m+t^2)+\operatorname{Ai}'(2^{1/3}x+m+t^2)\right]$$

$$B_m(x,y)=\operatorname{Ai}(x+y+m)$$

Thm [\(MorenoFlores-Q-Remenik June11\)](#) jt density of $\operatorname{argmax}_y\{\operatorname{Ai}_2(y) - \frac{y^2}{2}\}$ and $\max_y\{\operatorname{Ai}_2(y) - \frac{y^2}{2}\}$ is

$$f(t,m)=\gamma(t,4^{1/3}m)F_{\operatorname{GOE}}(4^{1/3}m)$$

$$\gamma(t,m)=2^{1/3}\langle\psi_{-t,4^{-1/3}m},(I-P_0B_mP_0)^{-1}\psi_{t,4^{-1/3}m}\rangle_{L^2(0,\infty)}$$

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[\(Schehr Mar12, Baik-Liechty-Schehr May12\)](#)

$$\gamma(t,m)=\frac{16}{\pi^2}\langle h_{4^{2/3}t},h_{-4^{2/3}t}\rangle_{L^2(m,\infty)}$$

$$h_t(x)=\int_0^\infty \zeta \Phi_2(\zeta,x)e^{-t\zeta^2}d\zeta$$

Φ_2 is the solution of one of the Lax equations for Painlevé II
 $q''=2q^3+tq$

Thm (MorenoFlores-Q-Remenik June11) jt density of $\operatorname{argmax}_y \{ \operatorname{Ai}_2(y) - \frac{y^2}{2} \}$
and $\max_y \{ \operatorname{Ai}_2(y) - \frac{y^2}{2} \}$ is

$$f(t, m) = \gamma(t, 4^{1/3}m) F_{\text{GOE}}(4^{1/3}m)$$

$$\gamma(t, m) = 2^{1/3} \langle \psi_{-t, 4^{-1/3}m}, (I - P_0 B_m P_0)^{-1} \psi_{t, 4^{-1/3}m} \rangle_{L^2(0, \infty)}$$

$$\psi_{t,m}(x) = 2e^{2^{1/3}xt} \left[t \operatorname{Ai}(2^{1/3}x + m + t^2) + \operatorname{Ai}'(2^{1/3}x + m + t^2) \right]$$

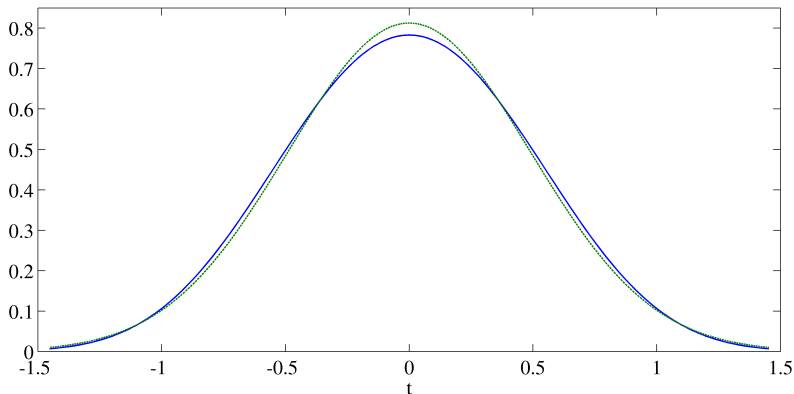
$$B_m(x, y) = \operatorname{Ai}(x + y + m)$$

(Schehr Mar12, Baik-Liechty-Schehr May12)

$$\gamma(t, m) = \frac{16}{\pi^2} \langle h_{4^{2/3}t}, h_{-4^{2/3}t} \rangle_{L^2(m, \infty)}$$

$$h_t(x) = \int_0^\infty \zeta \Phi_2(\zeta, x) e^{-t\zeta^2} d\zeta$$

Φ_2 is the solution of one of the Lax equations for Painlevé II
 $q'' = 2q^3 + tq$ (Dotsenko Sept12) Yet another formula, via replicas.
Similar, but still to prove that it coincides.



Polymer endpoint distribution (blue line) with Gaussian for comparison (dotted green line). Excess kurtosis $\frac{E[T^4]}{E[T^2]^2} - 3 = -0.2374$. Computation by Daniel Remenik. Surprisingly, Fredholm determinants *better* for numerics than Painleve (Bornemann).

HalpinHealy-Zhang conjecture: Tails decay like $e^{-c|x|^3}$

$$\text{Remenik-Q Mar12} \quad e^{-c|x|^3} \leq P(\mathcal{T} > x) \leq e^{-\frac{4}{3}|x|^3}$$

$$\text{Baik-Liechty-Schehr May12} \quad P(\mathcal{T} > x) \sim e^{-\frac{4}{3}|x|^3}$$

Complete picture for geometric last passage percolation

$\{w(i, j)\}_{i \in \mathbb{Z}^+, j \in \mathbb{Z}}$ iid geometric

i.e. $P(w(i, j) = m) = q(1 - q)^m$ for $m \geq 0$

last passage time is $\max_{r.w. \text{ paths } x(\cdot) \text{ starting at } 0} \sum_{i=0}^n w(i, x(i))$

$x(n)$ is the location at time n of the maximizer

Theorem (Johansson 02 + MorenoFlores-Q-Remenik 11)

$$P(x(n) \leq xn^{2/3}) \rightarrow \int_{-\infty}^x dt \int_{-\infty}^{\infty} dm \gamma(t, 4^{1/3}m) F_{\text{GOE}}(4^{1/3}m)$$

$$\gamma(t, m) = 2^{1/3} \langle \psi_{-t, 4^{-1/3}m}, (I - P_0 B_m P_0)^{-1} \psi_{t, 4^{-1/3}m} \rangle_{L^2(0, \infty)}$$

$$\psi_{t,m}(x) = 2e^{2^{1/3}xt} \left[t \text{Ai}(2^{1/3}x + m + t^2) + \text{Ai}'(2^{1/3}x + m + t^2) \right]$$

$$B_m(x, y) = \text{Ai}(x + y + m)$$

There is a similar formula for the Airy_1 process ([Q, Remenik 12](#))

$$P(\mathcal{A}_1(t) \leq g(t) \text{ for } t \in [\ell, r]) = \det \left(I - B_0 + \Lambda_{[\ell, r]}^g e^{-(r-\ell)\Delta} B_0 \right)_{L^2(\mathbb{R})}.$$

$$\Lambda_{[\ell, r]}^g(x, y) = \frac{e^{-(x-y)^2/4(r-\ell)}}{\sqrt{4\pi(r-\ell)}} P_{B(\ell)=x, B(r)=y}(B(s) \leq g(s) \text{ on } [\ell, r]),$$

$$B_0(x, y) = \text{Ai}(x + y)$$

We can use this to prove Airy_1 and Airy_2 are Hölder $1/2$ - and locally Brownian ([for \$\text{Airy}_2\$ Hagg 08](#))

Exercise in frustration: Try to prove continuity using extended kernel formula

Spatial persistence probabilities

$$P(\mathcal{A}(t) \leq m, 0 \leq t \leq L) \sim e^{-\kappa_- L}$$

Takeuchi measures $\kappa_-(\mathcal{A}_1) \approx 3.2 \pm 0.2$, $\kappa_-(\mathcal{A}_2) \approx 0.90 \pm 0.6$

Just need asymptotics on our determinantal formulas (trying with Schehr). Ferrari and Frings Sep12 numerically compute

$$\kappa_-(\mathcal{A}_1) \approx 2.9$$

Intriguingly, Takeuchi finds that $\kappa_+(\mathcal{A}_1) = \kappa_-(\mathcal{A}_1)$ where

$$P(\mathcal{A}(t) \geq m, 0 \leq t \leq L) \sim e^{-\kappa_+ L}$$

Question: Is there a way to compute κ_+ for the Airy processes?