UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS

MAT 235 Y - CALCULUS FOR PHYSICAL AND LIFE SCIENCES
FALL-WINTER 1995-96

TEST #1. NOVEMBER 14, 1995

NAME: ______________________, ______________ STUDENT No: ________________

(Family name. Please PRINT.) (Given name.)

INSTRUCTIONS: This test consists of six questions. The value of each question is indicated (in brackets) by the question number. Total marks: 45.
Show all your work in all questions. Give your answers in the space provided. Use both sides of the paper, if necessary. Do not tear out any pages.
No calculators or any other aids are permitted. This test is worth 15% of your course grade. Keep your student card visible on your table. Time allowed: 2 hours.

1. Let L_1 denote the line with symmetric equations $\frac{x-1}{3}=\frac{y}{-4}=\frac{z+2}{2}$, and let L_2 denote the line that passes through $P=(-2,1,0)$ and is parallel to L_1.
   a) (4 marks) Find parametric equations of the line L_2. Determine the coordinates of the point at which the line L_2 intersects the XZ-plane.
   b) (4 marks) Find an equation of the plane that contains both lines L_1 and L_2.

(a) Symmetric equations of L_2 are $\frac{x+2}{3} = \frac{y-1}{-4} = \frac{z}{2}$

so parametric equations of L_2 are $x = 3t-2$ $y = -4t+1$ $z = 2t$

so L_2 intersects XZ plane when $t = \frac{1}{4}$

so coords of point of intersection are $\left(3\cdot\frac{1}{4}-2,\ 0,\ 2\cdot\frac{1}{4}\right)$
$= \left(-\frac{5}{4},\ 0,\ \frac{1}{2}\right)$

(b) We can find an equation of the plane if we have a point on the plane and a normal vector. $(-2,1,0)$ is a point on the plane, and we can get a normal vector by taking the cross product of two vectors parallel to the plane.
One vector parallel to the plane is $(3,-4,2)$
and another is $(-2,1,0)-(1,0,-2) = (-3,1,2)$
Thus, a normal vector is $(3,-4,2)\times(-3,1,2)$
$=(-10,-12,-9)$, so an equation of the plane is

$-10(x+2) - 12(y-1) - 9z = 0$

2. Given the plane $x + 2y = 1$ and the surface $z = 3 + x - y^2$.

a) (3 marks) Find a parametrization of the curve that is the intersection of the plane and the surface.

b) (4 marks) Find all points of the curve at which the tangent vector is horizontal. (That is: points of the curve at which the z-component of the tangent vector is 0.)

(a) A point in the intersection of the plane and the surface satisfies $x + 2y = 1$ and $z = 3 + x - y^2$ and hence also satisfies: $x = 1 - 2y$

$$z = 3 + (1 - 2y) - y^2 = 4 - 2y - y^2$$

So the curve can be parameterized as follows:

$x = 1 - 2t \quad y = t \quad z = 4 - 2t - t^2$

(b) The tangent vector is $(-2, 1, -2-2t)$

This has 0 z-component iff $t = -1$,

and $t = -1$ gives the point $(3, -1, 5)$ on the curve.

3. Given the curve $\mathbf{r}(t) = (t^2, 2t, \ln t)$, $t>0$.

a) (4 marks) Find the arclength of this curve between the points corresponding to t=1 and t=2.

b) (4 marks) Find the curvature of this curve at t=1.

(a) $$\int_1^2 \sqrt{(2t)^2 + (2)^2 + \left(\tfrac{1}{t}\right)^2}\, dt$$

$$= \int_1^2 \sqrt{4t^2 + 4 + \tfrac{1}{t^2}}\, dt = \int_1^2 \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}}\, dt$$

$$= \int_1^2 \sqrt{\frac{(2t^2+1)^2}{t^2}}\, dt = \int_1^2 \frac{2t^2+1}{t}\, dt$$

$$= \int_1^2 2t\, dt + \int_1^2 \frac{1}{t}\, dt$$

$$= \left[t^2\right]_1^2 + \left[\ln t\right]_1^2 = 4 - 1 + \ln 2 - \ln 1$$
$$= 3 + \ln 2$$

(b) Tangent vector is $\left(2t, 2, \frac{1}{t}\right)$

and unit tangent vector is $\dfrac{1}{\sqrt{(2t)^2 + 2^2 + \left(\frac{1}{t}\right)^2}}\left(2t, 2, \frac{1}{t}\right)$.

$$= \frac{t}{2t^2+1}\left(2t, 2, \frac{1}{t}\right) = \left(1 - \frac{1}{2t^2+1}, \frac{2t}{2t^2+1}, \frac{1}{2t^2+1}\right)$$

Differentiating this with respect to t gives $\left(\dfrac{4t}{(2t^2+1)^2}, \dfrac{2(2t^2+1)-(2t)(4t)}{(2t^2+1)^2}, \dfrac{-4t}{(2t^2+1)^2}\right)$

To find curvature at $t=1$, evaluate the norm of this at $t=1$ and multiply by $\dfrac{1}{\sqrt{(2t)^2 + 2^2 + \left(\frac{1}{t}\right)^2}}$ also evaluated at $t=1$.

We get $\frac{1}{[illegible]} \cdot \sqrt{\left(\frac{4}{[illegible]}\right)^2 + \left(-\frac{2}{[illegible]}\right)^2 + \left(\frac{-4}{[illegible]}\right)^2} = \frac{2}{[illegible]}$