4. Consider the function $g(x,y)=\frac{x^2y}{x^4+y^2}$, which is defined for every point $(x,y)\neq(0,0)$.

a) (3 marks) What is the limit of the function g as (x,y) approaches (0,0) along the parabola $y=ax^2$?

b) (3 marks) Does $\lim_{(x,y)\to(0,0)} g(x,y)$ exist? Why or why not?

(a) As x, y approaches 0 along the parabola $y=ax^2$ we have

$$g(x,y)=\frac{x^2(ax^2)}{x^4+(ax^2)^2}=\frac{ax^4}{x^4+a^2x^4}$$

$$=\frac{ax^4}{(1+a^2)x^4}=\frac{a}{1+a^2}.$$

(b) $\lim_{(x,y)\to(0,0)} g(x,y)$ does not exist because as we approach (0,0) along different parabolas of the form $y=ax^2$, $g(x,y)$ does not always approach the same value, eg if $a=1$ then g approaches $\frac{1}{2}$ if $a=2$, g approaches $\frac{2}{5}\neq\frac{1}{2}$.

5. a) (4 marks) Suppose that the equation $xy\sin(x+z) + y^2e^z = 9$ is used to define z as an implicit function of x and y. Compute $\frac{\partial z}{\partial y}$ at $P=(\pi,3,0)$.

b) (4 marks) Consider the plane that is tangent to the surface $x^2 + xy^2 - yz^2 + 8z = -4$ at $x=2$, $y=0$, $z=-1$. Does the point (0,0,0) belong to this plane? Why or why not?

(a) Differentiating both sides of the equation $xy\sin(x+z) + y^2e^z = 9$ with respect to y yields

$$x\sin(x+z) + xy\cos(x+z)\frac{\partial z}{\partial y} + 2ye^z + y^2e^z\frac{\partial z}{\partial y} = 0$$

$$\left(xy\cos(x+z) + y^2e^z\right)\frac{\partial z}{\partial y} = -x\sin(x+z) - 2ye^z$$

$$\frac{\partial z}{\partial y} = \frac{-x\sin(x+z) - 2ye^z}{xy\cos(x+z) + y^2e^z}$$

So at $(\pi,3,0)$ we have $\frac{\partial z}{\partial y} = \frac{-6}{-3\pi+9} = \frac{2}{\pi-3}$.

(b) This is the surface $F(x,y,z) = 0$ where $F(x,y,z) = x^2 + xy^2 - yz^2 + 8z + 4$.

The gradient vector ∇F is $(2x+y^2, 2xy - z^2, -2yz+8)$ which is perpendicular to the surface. At $x=2, y=0, z=-1$ this is equal to $(4,-1,8)$ so the equation of the tangent plane is $4(x-2) - 1(y-0) + 8(z+1) = 0$ or $4x - y + 8z = 0$. Since this equation is satisfied by $x=0, y=0, z=0$, we conclude that the point $(0,0,0)$ does belong to the tangent plane.

6. Suppose that the formula $T = 75 - 2x^2 + y^2 - z^2$ gives the temperature (in ^{0}C) at any point (x,y,z).

a) (4 marks) Compute the directional derivative of the function T at the point P=(2,-1,1), in the direction of the vector **v**=(4,-3,12).

b) (4 marks) Find the unit vector indicating the direction in which T decreases most rapidly at the point P. What is the value of the minimal directional derivative of T at the point P?

(a) Gradient vector is $\nabla T = (-4x, 2y, -2z)$.

At P, this is $(-8, -2, -2)$.

Unit vector in direction of **v** is $\frac{1}{\sqrt{4^2+(-3)^2+12^2}}(4, -3, 12)$

$= \left(\frac{4}{13}, \frac{-3}{13}, \frac{12}{13}\right)$

Thus, directional derivative is $(-8,-2,-2)\cdot\left(\frac{4}{13}, \frac{-3}{13}, \frac{12}{13}\right)$

$= -\frac{50}{13}$

(b) Direction in which T decreases most rapidly is that of $-\nabla T(P)$, i.e. $(8, 2, 2)$. The unit vector in this direction is $\frac{1}{\sqrt{8^2+2^2+2^2}}(8, 2, 2) = \frac{1}{6\sqrt{2}}(8, 2, 2)$

$= \left(\frac{2\sqrt{2}}{3}, \frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}\right)$.

Thus the minimal directional derivative of T at P is $(-8, -2, -2)\cdot\left(\frac{2\sqrt{2}}{3}, \frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{6}\right)$

$= -6\sqrt{2}$.