5. SAMPLE QUESTIONS FROM PREVIOUS TEST #2 PAPERS.

1. Let $f(x,y)=x^2-xy-y^2$.
   a) (6 marks) Compute the directional derivative $D_{\mathbf{u}_1}f(2,1)$, where $\mathbf{u}_1=(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2})$.
   b) (4 marks) Find all unit vectors $\mathbf{u}_2$ for which $D_{\mathbf{u}_2}f(2,1)=0$.
   c) (5 marks) Find the unit vector $\mathbf{u}_3$ for which $D_{\mathbf{u}_3}f(2,1)$ attains its maximum value and the unit vector $\mathbf{u}_4$ for which $D_{\mathbf{u}_4}f(2,1)$ attains its minimum value.
2. Let S denote the surface given by the equation $x^2+y^2+2z^2-xy+xz+yz-z=6$.
   a) (8 marks) Find an equation of the plane that is tangent to the surface S at the point $(1,1,1)$.
   b) (7 marks) At what points of the surface S is the tangent plane parallel to the xy-plane ?
3. a) (10 marks) Consider the function $f(x,t)=\frac{1}{\sqrt{t}}e^{-u}$, where $u=\frac{(x-b)^2}{a^2t}$, $a\neq 0$, $t>0$. Simplify the expression $a^2\frac{\partial^2 f}{\partial x^2}(x,t)-4\frac{\partial f}{\partial t}(x,t)$.
   b) (10 marks) Show that $g(x,y,z)=\frac{r}{e^r}$, where $r=\sqrt{x^2+y^2+z^2}$ satisfies the differential equation $x\frac{\partial g}{\partial x}+y\frac{\partial g}{\partial y}+z\frac{\partial g}{\partial z}=(1-r)g$.
4. Indicate in each of the following cases whether the given statement is true or false. Give a brief and clear explanation for each of your answers.
   d) (3 marks) Any level surface of the function $f(x,y,z)=x^2+y^2-z^2$ is either a cone or a hyperboloid of one sheet.
   b) (3 marks) The function $f(x,y)=\begin{cases}\frac{x-y}{x+y} & \text{if } y\neq -x\\ 1 & \text{if } y=-x\end{cases}$ is continuous at $(0,0)$.
   c) (3 marks) Let $f(x,y)=x^{xy}$. Then $f_{xx}(1,1)=2$.
   d) (3 marks) Let y be implicitly defined as a function of x by the equation $(x^2+y^2)^2=3x^2y+y$, then when $x=1$ and $y=1$, the value of dy/dx is $-1/2$.

(Please turn over)

5. Let w denote a function of the variables u and v. If the change of variables $u = ax + y$ and $v = x + by$ is made, then w can also be regarded as a function of x and y.

a) (5 marks) Express $\frac{\partial^2 w}{\partial x \partial y}$ in terms of $\frac{\partial^2 w}{\partial u^2}$, $\frac{\partial^2 w}{\partial u \partial v}$ and $\frac{\partial^2 w}{\partial v^2}$.

b) (5marks) Suppose that the function w satisfies the equation $\frac{\partial^2 w}{\partial u^2} + 2\frac{\partial^2 w}{\partial u \partial v} - 8\frac{\partial^2 w}{\partial v^2} = 0$. Find suitable values of a and b for which this given equation can be converted to the simpler form $\frac{\partial^2 w}{\partial x \partial y} = 0$.

c) (5 marks) Can you guess the general form of the function w of the variables x and y which satisfies the equation $\frac{\partial^2 w}{\partial x \partial y} = 0$? If so, then state the general form of the function $w(x, y)$.

6. a) (10 marks) Use differentials to approximate the value of the function $f(x, y) = (x^3 + xy^2 - 6y)^{2/3}$ at the point $(1.99, 0.02)$.

b) (10 marks) Let S be the surface given by the equation $x^2 + xz = y^2 + z^2 - 4y + 5z$. Determine the coordinates of all points on the surface S (if any), at which the tangent plane to S is parallel to the plane $y = 0$.

7. (20 marks) A rectangular box is placed inside the ellipsoid $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$ with sides parallel to the axes. What dimensions will give the box with the maximum volume?

8. Given the function $f(x, y) = xy + 24 \ln x - 3y^2/2 - 18y$, where $x > 0$.

a) (5 marks) Find all the critical points of this function.

b) (5 marks) Use the second-derivative test to classify each of the critical points found in part (a) as a saddle point, a local maximum or a local minimum.

c) (10 marks) Find the absolute maximum and the absolute minimum values of the given function f over the triangle $1 \le x \le 3$, $2x \le y \le 6$.

9. (15 marks) Find the maximum and minimum values of the function $f(x, y) = x - 2y$, subject to the constraint $x^2 - 6x + 2y^2 = 39$.

10. a) (7 marks) Evaluate $\iint_R y \, dA$, where R is the region $0 \le x \le y - y^2$.

b) (8 marks) Evaluate the integral $\int_0^4 \int_{\sqrt{x}}^2 \frac{1}{\sqrt{1 + y^3}} \, dy \, dx$.

11. (10 marks) Compute the volume of the region above the triangle with vertices $(0, 0)$, $(1, 1)$ and $(0, 2)$ and under the surface $z = 9 - x^2 - y^2$.