UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS
MAT 235 Y - CALCULUS FOR PHYSICAL AND LIFE SCIENCE II
TEST #2. JANUARY 14, 2003

INSTRUCTIONS: Write your name and your student number on the front page of each of your examination booklets. Show all your work in all questions. Use both sides of the papers, if necessary. Do not tear out any pages. Do not use pencils. Only pen written answers will be considered for remarking. No calculators or any other aids are permitted. This test is worth 20% of your course grade. Duration: 1 hour and 50 minutes.

1. Consider the function $f(x,y) = \begin{cases} \dfrac{x^3+3y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

   a) (5 marks) Show that this function is not continuous at $(0,0)$.

   b) (5 marks) Compute $\dfrac{\partial f}{\partial x}(0,0)$.

2. a) (10 marks) Find an equation for the plane tangent to the surface $x(y+z) + \ln(x^2+y^2-z^2) = 2$ at the point $(1,1,1)$.

   b) (10 marks) In what direction is the function $f(x,y,z) = 3 - x^2 z \cos(yz)$ decreasing most rapidly from the point $(3,-2,0)$ and what is the value of the directional derivative in this direction ?

3. (10 marks) Let $u = \dfrac{y-5}{2+4x+y}$, where $x = 1 - r - \sqrt{5+r-s}$ and $y = s\sqrt{5+r-s}$.

   Compute $\dfrac{\partial u}{\partial s}$ and evaluate it at $(r,s) = (1,2)$.

4. (15 marks) Find and classify the critical points of the function $f(x,y) = (2xy - y - x^2 - 2x + 2)e^y$.

5. (10 marks) Maximize the function $f(x,y,z) = xy^2z^3$ subject to the constraint $x^2+y^2+z^2 = 6$.

6. (10 marks) Evaluate $\iint_R (2y - x^2)\,dA$ where R is the region lying between the parabola $y = x^2$ and the line $y = 2x$.

7. (10 marks) Evaluate the integral $\int_{-1}^{1}\int_{|x|}^{1} \dfrac{6x^2}{\sqrt{4+5y^4}}\,dy\,dx$ by reversing the order of integration.

8. (15 marks) Compute the volume of the solid under the surface $z = \sin(\pi\sqrt{x^2+y^2})$ and above the region $R = \{(x,y) \mid 1 \le 4(x^2+y^2) \le 4, xy \ge 0\}$.

MODEL SOLUTIONS

① a) If (x,y) approaches $(0,0)$ along the path $x=0$, then $f(x,y)=f(0,y)=3$ (if $y\neq 0$) and $\lim_{y\to 0} f(0,y)=3$.
If (x,y) approaches $(0,0)$ along the path $y=x$, then $f(x,x)=\frac{1}{2}(x+3)$ (if $x\neq 0$) and $\lim_{x\to 0} f(x,x)=\frac{3}{2}$.
So, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. Therefore $f(x,y)$ is not continuous at $(0,0)$.

b) Notice that, for all $x\in\mathbb{R}$, $f(x,0)=x$.
Then, for all $x\in\mathbb{R}$, $\frac{\partial f}{\partial x}(x,0)=1$, and $\frac{\partial f}{\partial x}(0,0)=1$.
(Or $\frac{\partial f}{\partial x}(0,0)=\lim_{h\to 0}\dfrac{\frac{h^3+0}{h^2+0}-0}{h}=\lim_{h\to 0}\frac{h}{h}=1$)

② a) Let $g(x,y,z)=x(y+z)+\ln(x^2+y^2-z^2)$
Then $\nabla g(x,y,z)=\left(y+z+\frac{2x}{x^2+y^2-z^2},\ x+\frac{2y}{x^2+y^2-z^2},\ x-\frac{2z}{x^2+y^2-z^2}\right)$

$\nabla g(1,1,1)=(4,3,-1)$. And an equation for the tangent plane is $4x+3y-z=6$.

b) $\nabla f(x,y,z)=(-2xz\cos(yz),\ x^2z^2\sin(yz),\ -x^2\cos(yz)+x^2yz\sin(yz))$

$\nabla f(3,-2,0)=(0,0,-9)$
So, the direction of most rapid decrease is given by the vector $(0,0,1)$, and the value of the directional derivative in this direction is -9.

(3) $\dfrac{\partial u}{\partial x} = -\dfrac{4(y-5)}{(2+4x+y)^2}$ $\qquad \dfrac{\partial u}{\partial y} = \dfrac{2+4x+y-(y-5)}{(2+4x+y)^2} = \dfrac{2+4x+5}{(2+4x+y)^2}$

$\dfrac{\partial x}{\partial s} = \dfrac{1}{2\sqrt{5+r-s}}$ $\qquad \dfrac{\partial y}{\partial s} = \sqrt{5+r-s} - \dfrac{s}{2\sqrt{5+r-s}}$

At $r=1$, $s=2$: $x=-2$, $y=4$, and

$\dfrac{\partial u}{\partial x} = \dfrac{4}{4} = 1$, $\dfrac{\partial u}{\partial y} = -\dfrac{1}{4}$, $\dfrac{\partial x}{\partial s} = \dfrac{1}{4}$, $\dfrac{\partial y}{\partial s} = \dfrac{3}{2}$

Then $\dfrac{\partial u}{\partial s} = (1)(+\frac{1}{4}) + (-\frac{1}{4})(\frac{3}{2}) = -\frac{1}{8}$

(4) $f_x = (2y-2x-2)e^y$ $\qquad f_y = (2xy - y - x^2 - 2x + 2 + 2x - 1)e^y$

$= (2xy - y - x^2 + 1)e^y$

For $f_x = 0 = f_y$: $y = x+1$, and $2x(x+1) - (x+1) - x^2 + 1 = 0$

$x^2 + x = 0$, $x = 0$ or $x = -1$

Critical points: $(0,1)$ and $(-1,0)$

$f_{xx} = -2e^y$ $\quad f_{xy} = (2y-2x-2+2)e^y = 2(y-x)e^y = f_{yx}$

$f_{yy} = (2xy - y - x^2 + 1 + 2x - 1)e^y = (2xy - y - x^2 + 2x)e^y$

At $(0,1)$: $f_{xx} = -2e < 0$ and $\det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{pmatrix} = \det\begin{pmatrix} -2e & 2e \\ 2e & -e\end{pmatrix}$

$= -2e^2 < 0$

Saddle point at $(0,1)$.

At $(-1,0)$: $f_{xx} = -2 < 0$ and $\det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{pmatrix} = \det\begin{pmatrix} -2 & 2 \\ 2 & -3\end{pmatrix}$

$= 2 > 0$

Local maximum at $(-1,0)$.

⑤ Let $g(x,y,z) = x^2 + y^2 + z^2$, then for $\nabla f = \lambda \nabla g$:

$y^2 z^3 \overset{①}{=} 2\lambda x$, $2xyz^3 \overset{②}{=} 2\lambda y$, and $3xy^2z^2 \overset{③}{=} 2\lambda z$

Notice that $x \neq 0$, $y \neq 0$, $z \neq 0$, otherwise $f = 0$, which can not be the maximum of f.

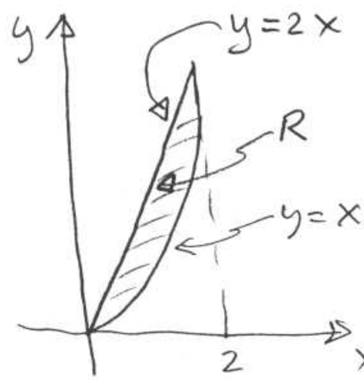
Then, from ②: $\lambda = xz^3$, and from ①: $y^2 = 2x^2$

also, from ③: $3y^2 = 2z^2$, and $z^2 = 3x^2$

So, $x^2 + y^2 + z^2 = 6x^2 = 6$, then $x^2 = 1$, $y^2 = 2$ $z^2 = 3$

That is $x = \pm 1$, $y = \pm\sqrt{2}$, $z = \pm\sqrt{3}$ and the maximum value of f is $(1)(\sqrt{2})^2(\sqrt{3})^3 = 6\sqrt{3}$.

⑥



$R: \begin{cases} 0 \le x \le 2 \\ x^2 \le y \le 2x \end{cases}$

$$\iint_R (2x - y^2)\, dA = \int_0^2 \int_{x^2}^{2x} (2y - x^2)\, dy\, dx$$

$$= \int_0^2 \left[y^2 - x^2 y \right]_{x^2}^{2x} dx$$

$$= \int_0^2 (4x^2 - 2x^3 - \cancel{x^4} + \cancel{x^4})\, dx$$

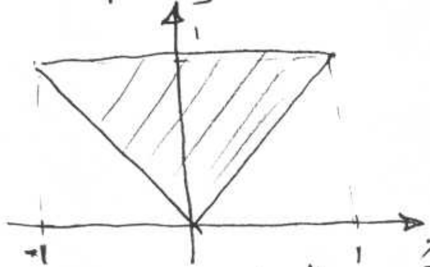
$$= \int_0^2 (4x^2 - 2x^3)\, dx$$

$$= \left[\frac{4}{3}x^3 - \frac{1}{2}x^4 \right]_0^2$$

$$= \frac{32}{3} - 8 = \frac{8}{3}$$

(7) The region of integration is

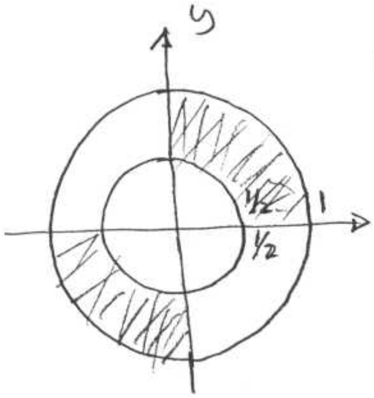
$$R: \begin{cases} -1 \le x \le 1 \\ |x| \le y \le 1 \end{cases} \quad \text{or} \quad R: \begin{cases} 0 \le y \le 1 \\ -y \le x \le y \end{cases}$$



$$\int_{-1}^{1}\int_{|x|}^{1} \frac{6x^2}{\sqrt{4+5y^4}}\,dy\,dx = \int_0^1\int_{-y}^{y} \frac{6x^2}{\sqrt{4+5y^4}}\,dx\,dy = \int_0^1 \left.\frac{2x^3}{\sqrt{4+5y^4}}\right]_{-y}^{y} dx$$

$$= \int_0^1 \frac{4y^3}{\sqrt{4+5y^4}}\,dy = \left.\frac{2}{5}\sqrt{4+5y^4}\,\right]_0^1 = \frac{2}{5}$$

(8)



In polar coord: $\frac{1}{2} \le r \le 1$

and $0 \le \theta \le \frac{\pi}{2}$ or $\pi \le \theta \le \frac{3\pi}{2}$

Then $V = 2\int_0^{\pi/2}\int_{1/2}^{1} r\sin(\pi r)\,dr\,d\theta$

$$= 2\left[\theta\right]_0^{\pi/2}\left[-\frac{r}{\pi}\cos(\pi r) + \frac{1}{\pi^2}\sin(\pi r)\right]_{1/2}^{1}$$

$$= 2\left(\frac{\pi}{2}\right)\left(\frac{1}{\pi} + 0 + 0 - \frac{1}{\pi^2}\right)$$

$$= \frac{\pi - 1}{\pi}$$