UNIVERSITY OF TORONTO
DEPARTMENT OF MATHEMATICS
MAT 235 Y - CALCULUS FOR PHYSICAL AND LIFE SCIENCE II
TEST #3. MARCH 4, 2003

INSTRUCTIONS: Write your name and your student number on the front page of each of your examination booklets. Show all your work in all questions. Use both sides of the papers, if necessary. Do not tear out any pages. Do not use pencils. Only pen written answers will be considered for remarking. No calculators or any other aids are permitted. This test is worth 20% of your course grade. Duration: 1 hour and 50 minutes.

1. Consider the vector field $\mathbf{F}(x,y,z) = (y - 2xz)\,\mathbf{i} + x\,\mathbf{j} + (1 - x^2)\,\mathbf{k}$.

   a) (5 marks) Find a function f such that $\mathbf{F} = \nabla f$.

   b) (5 marks) Use this potential function f to evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the curve given by the parametrization $\mathbf{r}(t) = (2+t)\,\mathbf{i} + 2t^2\,\mathbf{j} + (1 - t^3)\,\mathbf{k}$, $0 \le t \le 1$.

2. (15 marks) Evaluate the line integral $\int_C (x + xy - 6z)\,ds$, where C is the curve given by the parametric equations $x = 2t$, $y = t^2$, $z = t^3/3$, $\sqrt{2} \le t \le 2$.

3. (15 marks) Evaluate the triple integral $\iiint_E \sqrt{x^2 + y^2 + z^2}\, dV$, where E is the solid bounded below by the conic surface $3z^2 = x^2 + y^2$, $z \ge 0$, and bounded above by the spherical surface $x^2 + y^2 + z^2 = 5$.

4. (15 marks) Calculate the area of the part of the surface $9z^2 = 2(x - y)^3$ that lies above the triangle with vertices $(0,0)$, $(3,0)$ and $(3,3)$.

5. (15 marks) Evaluate the double integral $\iint_\Omega (2x - y)^2 e^{4x^2 - y^2}\, dA$, where Ω is the parallelogram bounded by the straight lines $2x + y = 0$, $2x - y = 0$, $2x + y = 1$ and $2x - y = 2$.

6. (15 marks) Evaluate the line integral $\int_C (y^2 e^x - y\sqrt{y})\,dx + (2ye^x + 6x\sqrt{y})\,dy$, where the curve C is the circle $x^2 + (y-1)^2 = 1$ with the positive orientation.

   Hint: Apply Green's Theorem and use polar coordinates to evaluate the double integral.

7. (15 marks) Evaluate the triple integral $\iiint_E \frac{6y^2 z}{(1 + x^4)^2}\, dV$, where $E = \{ (x,y,z) \mid 0 \le y \le 2x, x^2 \le z \le 2 \}$.

MODEL SOLUTIONS

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① a) $\boxed{f = xy - x^2 z + z}$

b) $\int_C F\, dr = f(r(1)) - f(r(0))$

$= f(3,2,0) - f(2,0,1)$

$= 6 - (-4+1) = \boxed{9}$

② $\int_C (x + xy - 6z)\, ds = \int_{\sqrt{2}}^{2} (2t + 2t^3 - 2t^3)\sqrt{2^2 + (2t)^2 + (t^2)^2}\, dt$

$= \int_{\sqrt{2}}^{2} 2t\sqrt{4 + 4t^2 + t^4}\, dt$

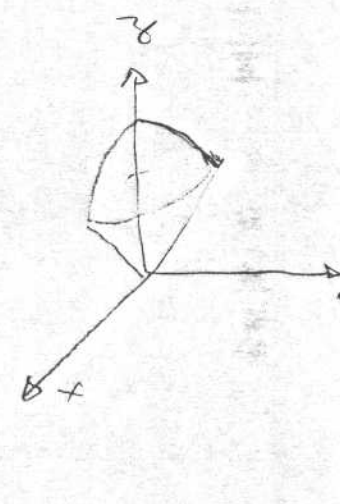
$= \int_{\sqrt{2}}^{2} 2t(t^2 + 2)\, dt$

$= \frac{t^4}{2} + 2t^2 \Big]_{\sqrt{2}}^{2}$

$= 8 + 8 - 2 - 4 = \boxed{10}$

③ Using spherical coordinates:

$$E \begin{cases} 0 \le \rho \le \sqrt{5} \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \frac{\pi}{3} \end{cases}$$



$\iiint_E \sqrt{x^2+y^2+z^2}\, dV = \int_0^{\sqrt{5}} \int_0^{2\pi} \int_0^{\pi/3} \rho^3 \sin\phi\, d\phi\, d\theta\, d\rho$

$= \left[\frac{1}{4}\rho^4\right]_0^{\sqrt{5}} \left[\theta\right]_0^{2\pi} \left[-\cos\phi\right]_0^{\pi/3}$

$= \left(\frac{25}{4}\right)(2\pi)\left(-\frac{1}{2} + 1\right)$

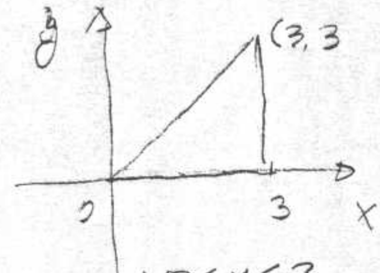
$= \boxed{\frac{25\pi}{4}}$

4

$$18 z z_x = 6(x-y)^2$$

$$z_x = \frac{(x-y)^2}{3z}$$

$$z_y = -\frac{(x-y)^2}{3z}$$



$$\begin{cases} 0 \le x \le 3 \\ 0 \le y \le x \end{cases}$$

$$A = \int_0^3 \int_0^x \sqrt{1 + \frac{(x-y)^4}{9z^2} + \frac{(x-y)^4}{9z^2}}\, dy\, dx$$

$$= \int_0^3 \int_0^x \sqrt{1 + \frac{x-y}{2} + \frac{x-y}{2}}\, dy\, dx$$

$$= \int_0^3 \int_0^x \sqrt{1 + x - y}\, dy\, dx$$

$$= \int_0^3 -\frac{2}{3}(1+x-y)^{3/2}\Big]_0^x\, dx$$

$$= \int_0^3 -\frac{2}{3}\left(1 - (1+x)^{3/2}\right) dx$$

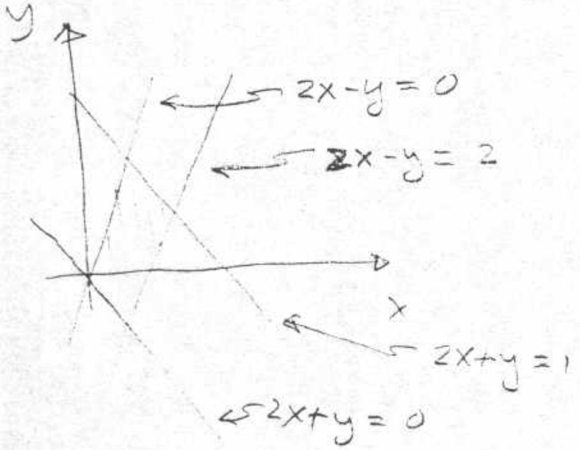
$$= -\frac{2}{3}\left(x - \frac{2}{5}(1+x)^{5/2}\right)\Big]_0^3$$

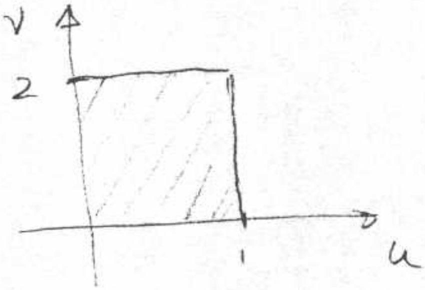
$$= -\frac{2}{3}\left(3 - \frac{2}{5}(32) - 0 + \frac{2}{5}\right)$$

$$= \left(-\frac{2}{3}\right)\left(\frac{15-64+2}{5}\right) = \boxed{\frac{94}{15}}$$

(5) Make $\begin{cases} u = 2x+y \\ v = 2x-y \end{cases}$ then $\begin{cases} x = \frac{1}{4}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases}$

$$\det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} 1/4 & 1/4 \\ 1/2 & -1/2 \end{pmatrix} = -\frac{1}{8} - \frac{1}{8} = -\frac{1}{4}$$





$$\iint_{\Omega} (2x-y)^2 e^{4x^2-y^2}\, dA = \int_0^2 \int_0^1 \frac{1}{4} v^2 e^{uv}\, du\, dv$$

$$= \int_0^2 \frac{1}{4} v e^{uv} \Big]_0^1 dv = \int_0^2 \frac{1}{4}(ve^v - v)\, dv$$

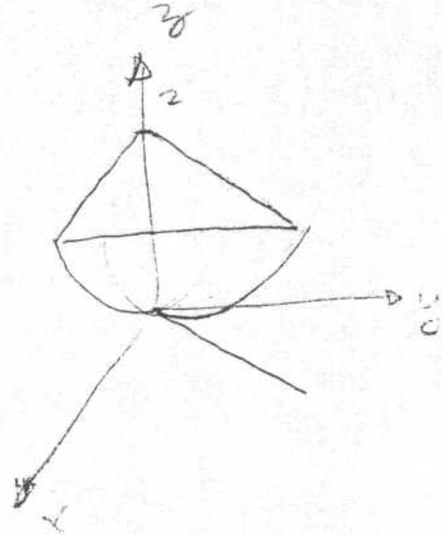
$$= \frac{1}{4}\left[ve^v - e^v - \frac{1}{2}v^2 \right]_0^2$$

$$= \frac{1}{4}\left[2e^2 - e^2 - 2 - 0 + 1 + 0 \right]$$

$$= \boxed{\frac{1}{4}(e^2 - 1)}$$

(7)

$$E: \begin{cases} 0 \le z \le 2 \\ 0 \le x \le \sqrt{z} \\ 0 \le y \le 2x \end{cases}$$



$$\iiint_E \frac{6y^2 z}{(1+x^4)^2}\, dV$$

$$= \int_0^2 \int_0^{\sqrt{z}} \int_0^{2x} \frac{6y^2 z}{(1+x^4)^2}\, dy\, dx\, dz$$

$$= \int_0^2 \int_0^{\sqrt{z}} \left. \frac{2y^3 z}{(1+x^4)^2} \right]_0^{2x} dx\, dz$$

$$= \int_0^2 \int_0^{\sqrt{z}} \frac{16x^3 z}{(1+x^4)^2}\, dx\, dz$$

$$= \int_0^2 \left. -\frac{4z}{1+x^4} \right]_0^{\sqrt{z}} dz$$

$$= \int_0^2 \left[-\frac{4z}{1+z^2} + 4z \right] dz$$

$$= \left[2z^2 - 2\ln(1+z^2) \right]_0^2$$

$$= \boxed{8 - 2\ln 5}$$