VOLUME, DIAMETER AND THE MINIMAL MASS OF A STATIONARY 1-CYCLE

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Abstract. In this paper we present upper bounds on the minimal mass of a non-trivial stationary 1-cycle. The results that we obtain are valid for all closed Riemannian manifolds. The first result is that the minimal mass of a stationary 1-cycle on a closed n-dimensional Riemannian manifold M^n is bounded from above by $\frac{(n+2)!d}{4}$, where d is the diameter of a manifold M^n. The second result is that the minimal mass of a stationary 1-cycle on a closed Riemannian manifold M^n is bounded from above by $(n+2)!FillRad(M^n) \leq (n+2)!(n+1)n^n\sqrt{(n+1)!}(vol(M^n))^{1/n}$, where $FillRad(M^n)$ is the filling radius of a manifold, and $vol(M^n)$ is its volume.

1. Introduction.

Let $l(M^n)$ denote the length of a shortest closed geodesic on a Riemannian manifold M^n. In 1983 M. Gromov asked whether there exists a constant $c(n)$ such that $l(M^n) \leq c(n)(vol(M^n))^{\frac{1}{n}}$, (see [G], p. 135). This problem also appeared as Problem 87 in a list of open problems in Differential Geometry composed by S.-T. Yau ([Y], p. 689, or [SY], p. 297). In the same spirit it might be interesting to know whether there exists $\tilde{c}(n)$ such that $l(M^n) \leq \tilde{c}(n)d$, where d denotes the diameter of M^n.

At the moment the only known explicit upper bounds for the length of a shortest closed geodesic on an arbitrary closed Riemannian manifold M^n are the estimates that were found in [NR]; see also the earlier paper [R]. Those estimates, however, use information about either the sectional curvature or the injectivity radius of the manifold.

In the present paper we prove the existence of a stationary 1-cycle such that its mass satisfies these inequalities. In fact, our proofs demonstrate the existence of a stationary 1-cycle of a special type which we will call *strongly stationary* such that its mass satisfies these inequalities. Strongly stationary 1-cycles are defined as follows.

Definition 1. *A strongly stationary 1-cycle consists of finitely many points* $p_1, \dots, p_l$ *and a finite collection of (not necessarily distinct) geodesic segments that start and end at these points so that the following two conditions are satisfied for each point* p_i *,* $i = 1, 2, \dots l$ *:*
1) (Cycle condition) The number of geodesic segments meeting at p_i *is even. (Here each geodesic loop based at* p_i *is counted twice; multiple geodesic segments are counted with their multiplicities.)*
2) (Stationarity Condition) The sum of unit vectors in $T_{p_i}M^n$ *tangent to all geodesic segments meeting at* p_i *(counted with their multiplicities) is equal to zero. Here each tangent vector is directed from* p_i *.*

In other words, stationary 1-cycles are immersed finite multigraphs such that all edges are realized by geodesic segments, each vertex has an even degree, and for each vertex the stationarity condition 2) holds.

Each closed geodesic can be regarded as a strongly stationary 1-cycle. A geodesic loop is a strongly stationary 1-cycle if and only if it is a closed geodesic. Some simple examples of strongly stationary 1-cycles that do not correspond to closed geodesics are shown on Fig. 2. Informally speaking, strongly stationary 1-cycles can be viewed as a homological analog of closed geodesics.

In section 2 we are going to give a slightly different but equivalent definition of strongly stationary 1-cycles, which will explain the term "strongly stationary", (see def. 4).

Definition 1 can be compared with the definition of *geodesic nets* in the paper of J. Hass and F. Morgan [HM]. Geodesic nets satisfy the stationarity condition but need not satisfy the cycle condition. However, Hass and Morgan require that all geodesics forming a geodesic net must be embedded and cannot intersect each other. They also do not allow multiple geodesics with the same endpoints.

The *mass* or *length* of a strongly stationary 1-cycle is defined as the sum of lengths of all its geodesic segments counted with their multiplicities. (The terms mass and length will be used interchangeably.) A strongly stationary 1-cycle is called *non-trivial* if its mass is not equal to zero. (The geodesic segments in the definition of a strongly stationary 1-cycle are allowed to be trivial geodesics.)

We also obtain an explicit upper bound for the total number of all geodesic segments (counted with their multiplicities) in a non-trivial strongly stationary 1-cycle of mass not exceeding $\tilde{c}(n)diam(M^n)$ (or $c(n)vol(M^n)^{\frac{1}{n}}$) in terms of n.

Of course, our estimates would give the estimates on the length of a shortest closed geodesic if the strongly stationary 1-cycle we obtain is realized by a closed geodesic. And, in fact, when M^n is diffeomorphic to the 2-dimensional sphere, this technique produces a closed geodesic, as was observed by J. Pitts, E. Calabi and J. Cao, (see [ClCo]). This fact enabled us to obtain estimates for the length of a shortest closed geodesic on a manifold diffeomorphic to S^2, (see [NR1]) improving previously known results by C.B. Croke and M. Maeda, (see [C], [Ma]). Similar results were independently obtained by S. Sabourau in [S1]. Sabourau had also found curvature free upper bounds on the length of a shortest geodesic loop on a compact Riemannian manifold, (see [S2]). To compare the results of the present paper with this result of Sabourau note that for each point of M^n there are infinitely many geodesic loops based at this point. So the set of all geodesic loops on M^n is uncountable. However strongly stationary 1-cycles are critical points of a certain functional (see below). Therefore a standard argument implies that there are only countably many strongly stationary 1-cycles for a generic analytic metric on M^n.

We refer the reader to a survey written by C. B. Croke and M. Katz ([CK]) for an exposition of other curvature-free estimates in Riemannian geometry as well as of the theory of *systolic freedom* developed by I. Babenko, M. Katz, A. Suciu. (This theory enables one to prove that some plausibly looking curvature-free estimates in fact do not hold.)

The techniques that we use in the present paper were partially inspired by the geometric measure theory approach to the existence of minimal submanifolds developed by F. J. Almgren and J. Pitts, (see [P]). However, since we deal only with "nice" 1-dimensional cycles we find ourselves in a much more geometrical situation and do not need almost anything from the elaborate language and machinery of geometric measure theory. Sections 2-4 contain the adaptation of all the necessary results and ideas from GMT to our context. (Our main results will be proven in section 5.)

We also use an appropriate generalization of obstruction to an extension technique used by M. Gromov in [G] (see section 1.2 of [G] as well as the proof of Proposition on p. 136 of [G]), and in the case of the Theorem 2 below we use Gromov's upper bound for the filling radius in terms of volume.

Now we are going to state our main results.

Theorem 1. *Let M^n be a closed simply-connected Riemannian manifold of dimension n. Let $q(\leq n)$ denote the minimal dimension i such that $\pi_i(M^n)\neq 0$.*

Then there exists a non-trivial strongly stationary 1-cycle on M^n that consists of at most $\frac{(q+2)!}{2}$ geodesic segments such that its mass does not exceed $\frac{(q+2)!}{4}d$. If $q=2$ then there exists a strongly stationary 1-cycle of length $\leq 4d$ that is either a closed geodesic or consists of two geodesic loops emanating from the same point p.

Remark. It is well-known (and easy to prove) that if M^n is *not* simply-connected then there is a closed geodesic on M^n of length $\leq 2d$.

In order to state the next theorem we will need the following definitions and the following result of M. Gromov (see [G]).

Definition 2.A. *Let M^n be a manifold topologicaly imbedded into an arbitrary metric space X. Then its filling radius, denoted $FillRad(M\subset X)$, is the infimum of $\epsilon>0$, such that M^n bounds in the ϵ-neighborhood $N_\epsilon(M^n)$, i.e. homomorphism $H_n(M^n)\longrightarrow H_n(N_\epsilon(M^n))$ induced by the inclusion map vanishes, where $H_n(M)$ denotes the singular homology group of dimension n with coefficients in $\mathbb{Z}$, when M is orientable, and with coefficients in $\mathbb{Z}_2$, when M is not orientable.*

Definition 2.B. *Let M^n be an abstract manifold. Then its filling radius, denoted $FillRadM^n$ will be $FillRad(M\subset X)$, where $X=L^\infty(M)$, i.e. the Banach space of bounded Borel functions f on M^n and the imbedding of M^n into X is the map that assigns to each point p of M^n the distance function $p\longrightarrow f_p=d(p,q)$.*

Theorem A ([G]). *Let M^n be a closed connected Riemannian manifold. Then $FillRadM^n\leq (n+1)n^n(n+1)!^{\frac{1}{2}}(volM^n)^{\frac{1}{n}}$.*

Theorem 2. *Let M^n be a closed Riemannian manifold. Then there exists a non-trivial strongly stationary 1-cycle in M^n made of at most $(n+2)!/2$ geodesic segments such that its mass is bounded from above by $(n+2)!FillRadM^n\leq(n+2)!(n+1)(n+1)!^{\frac{1}{2}}n^n(vol(M^n))^{\frac{1}{n}}$.*

Remark. Note that M. Katz proved that the filling radius of a manifold M^n does not exceed $\frac{1}{3}diam(M^n)$ (and this estimate is exact), [K]. Therefore Theorem 2 immediately implies a slightly weaker version of Theorem 1 (with a worse value of the constant).

Now we will present the main ideas of the proof of Theorem 1. For the sake of simplicity, let us assume that $\pi_2(M) \neq \{0\}$. Let us begin with any non-contractible $f : S^2 \longrightarrow M$. Then $f_*([S^2])$ does not bound in M. Assume that the minimal length of a non-trivial strongly stationary 1-cycle is greater than $6d$. Then $l(M) > 6d$. We will find a 3-chain that has $f_*([S^2])$ as its boundary, thus obtaining a contradiction.

Assuming S^2 has a sufficiently fine triangulation, triangulate D^3 as a cone over S^2. To construct a 3-chain we will use the following "extension" procedure, which will be inductive to skeleta of D^3. We let the only additional 0-vertex of D^3, i.e. its center p, be mapped to an arbitrary point $\tilde{p}$ of M, and we will let the edges, i.e. line segments $[v_i, p]$ connecting vertices of the triangulation of S^2 with p, be mapped to minimal geodesic segments of length at most d that connect $\tilde{p}$ with the corresponding vertices $\tilde{v}_i = f(v_i)$.

Next we are going to extend to the 2-skeleton. We consider an arbitrary 2-simplex $[v_i, v_j, p]$ and notice that its boundary is mapped to a closed curve of length of at most $2d+\delta$, (that is, assuming that the diameter of each simplex is less than δ). Now, if δ is sufficiently small, so that $l(M) > 2d+\delta$ by our assumption, then each such curve can be contracted to a point without length increase, thus for each such curve we obtain a disc, and we can map each 2-simplex to the corresponding disc.

At the next step we want to "extend" to the 3-skeleton. At this stage we will use some basic Morse theory on the space of 1-cycles.

In order to "extend" to the 3-skeleton, consider an arbitrary 3-dimensional simplex. Consider its boundary that consists of four faces. We want to construct a loop in the space of 1-cycles that corresponds to this boundary. Here is how we do it. Each face corresponds to a line segment in ΛM, the space of continuous closed curves on M, that begins with a constant curve and ends with the curve that is the image of the boundary of the corresponding face. This line segment passes through curves of length less than or equal to $2d + \delta$. Now the main idea is to consider those line segments "together", thus obtaining a line segment in the space of cycles. This segment begins with a 1-cycle of length 0 (consisting of four points each counted with multiplicity three) and ends with a four properly oriented images of the boundaries of the corresponding faces. If we considered 1-cycles as functionals on the space of 1-forms (as it is customary in geometric measure theory), then the sum of the above boundaries would have constituted the zero cycle. So, we would have obtained a loop in the space of 1-cycles of length $\leq 6d+6\delta$. But here we consider 1-cycles that consist of parametrized segments, so no cancellation of oppositely directed segments would occur. Therefore we group $4\times 3=12$ parametrized segnents into 6 pairs of segments with the identical image but opposite orientations and shrink each pair to its midpoint. The resulting 12 points are then connected by some paths with original $4\times 3=12$ points, which will close the loop. Thus, we obtain a loop in the space of 1-cycles of length at most $6d+6\delta$.

We next use a Morse-theoretic lemma (see Proposition 4, below) to claim that either we can contract the loop to a point along cycles of length at most $6d+6\delta$, or there exists a "nice" stationary cycle below this loop (i.e. the length of the stationary 1-cycle does not exceed $6d+6\delta$). In the former case we can, indeed, contract each such loop to a point, and obtain corresponding 2-discs in the space of 1-cycles, which correspond to 3-chains on a manifold. Each of these 3-chains consists of all 1-cycles in the image of the corresponding 2-disc, (see section 4 for more details). Add those 3-chains and note that the boundary of the sum will be $f_*([S^2])$, thus, obtaining a contradiction. We can now let δ go to zero.

2. Basic definitions.

2.1 Spaces of 1-cycles that will be used in the present paper Now we would like to introduce some spaces of "nice" 1-cycles that are especially useful for our purposes. Following [ClCo] it is convenient to consider spaces of *parametrized* 1-cycles made of at most k closed curves: Define Γ_k as the space of all k-tuples $(\gamma_1,\ldots,\gamma_k)$ of Lipschitz maps of $[0,1]$ to M^n such that $\Sigma_{i=1}^k\gamma_i(0)=\Sigma_{i=1}^k\gamma_i(1)$. Endow Γ_k with the following metric topology: First, because of the Nash embedding theorem we can assume without any loss of generality that M^n is isometrically embedded into the Euclidean space R^N of a large dimension. Define the distance by the formula:

$$d((\alpha_1,\dots,\alpha_k),(\gamma_1,\dots,\gamma_k)) = \max_{i,t} d_{M^n}(\alpha_i(t),\gamma_i(t)) + \Sigma_{i=1}^k \sqrt{\int_0^1 |\alpha_i'(t)-\gamma_i'(t)|^2 dt}.$$

It is easy to see that the length functional

$$l((\gamma_1,\dots,\gamma_k)) = \Sigma_{i=1}^k l(\gamma_i)$$

is a continuous functional on this space. Observe that we can assign to each element A of Γ_k a bounded linear functional T_A on the space $\Omega_1(M^n)$ of 1-forms on M^n by the formula $T_A(\omega) = \int_A \omega = \Sigma_{i=1}^k \int_{\gamma_i} \omega$. Thus, we obtain the map $I : \Gamma = \cup_k \Gamma_k \longrightarrow (\Omega_1(M^n))^*$ to the dual space of the space of 1-forms. Denote the image of Γ under this map by $Z_1(M^n,\mathbb{Z})$. This space will be called *the space of non-parametrized 1-cycles on* M^n. Denote the image of Γ_k under I by $Z_{(k)}$. We will call $Z_{(k)}$ *the space of (non-parametrized) 1-cycles on* M^n *made of at most* k *closed curves.* Observe that Γ_k contains all collections of at most k suitably parametrized closed curves in M^n. (see Fig. 3 for examples of elements of Γ_2.) Therefore $Z_{(1)} \subset Z_{(2)} \subset Z_{(3)} \subset \dots$, and $\bigcup_{i=1}^\infty Z_{(i)} = Z_1(M^n,\mathbb{Z})$. Also, for any x let the subset of Γ_k formed by all $\gamma = (\gamma_1,\dots,\gamma_k)$ such that $l(\gamma) = \Sigma_{i=1}^k l(\gamma_i) \le x$ be denoted by Γ_k^x, and the image of Γ_k^x under I be denoted by $Z_{(k)}^x$. We will call $Z_{(k)}^x$ *the space of 1-cycles on* M^n *of length* $\le x$ *made of at most* k *curves.* Similarly, we will call elements of Γ_k *parametrized 1-cycles made of* k *curves*, k will be called *the order* of parametrized cycles from Γ_k, and elements of Γ_k^x will be called *parametrized 1-cycles of length* $\le x$ *made of* k *curves.*

The fundamental difference between spaces $Z_{(k)}$ and Γ_k is that if we go along a curve γ and then backtrack via the same curve, we obtain the trivial non-parametrized cycle, but a non-trivial parametrized cycle. This feature of non-parametrized cycles makes some of our constructions easier and more transparent when they are carried out for $Z_{(k)}$. Therefore we first explain how to perform some of our constructions in $Z_{(k)}$ before explaining how they can be done in Γ_k (which is what is actually needed for our purposes). In the present paper we will be using non-parametrized cycles only for illustrative purposes.

2.2 Strongly stationary 1-cycles

Let X be a smooth vector field on M^n. It determines a one-parameter group of diffeomorphisms $\Phi_X(t)$ of M^n. For any $\gamma \in \Gamma_k$ consider the one-parameter family of parametrized 1-cycles $\Phi_X(t)(\gamma)$. Now one can consider the function $L_{X,\gamma}(t)$ defined as the total length of k Lipschitz curves that together form $\Phi_X(t)(\gamma)$.

Definition 3. *Let $\gamma \in \Gamma_k$ be a parametrized 1-cycle. We say that γ is strongly stationary if it satisfies the following two conditions: 1) (Stationarity) For any smooth vector field X $\frac{dL_{X,\gamma}}{dt}(0) = 0$; and 2) For each $i = 1, \dots k$ γ_i is a geodesic. If γ is strongly stationary, then its image $I(\gamma)$ in $Z_{(k)}$ is called a stationary (non-parametrized) 1-cycle. For each $\gamma \in \Gamma_k$ and each smooth vector field X on M^n the value of the derivative of $L_{X,\gamma}$ at $t = 0$ will be called the first variation of the length of γ in the direction of X. We will denote the minimal length of a non-trivial strongly stationary parametrized 1-cycle in M^n by $\alpha(M^n)$.*

On the first glance it might seem that the condition 2) in this definition is extraneous. The following example shows that this is not so:

Example 1. Let $k = 1$. Assume that $\gamma = \gamma_1 \in \Gamma_1$ is a three-petal curve that consists of three geodesic loops emanating from the same point p. Consider three angles formed by pairs of tangent vectors for each petal (=geodesic loop) at p (see Fig. 2(b). As usual, we direct the tangent vectors from p.) Assume further that: 1) These three angles have equal values that are strictly less than $2\pi/3$; 2) The bisectors of these three angles lie in a plane in T_pM^n and form angles equal to $2\pi/3$ with each other. Then it is easy to see that γ satisfies the stationarity condition, but γ_1 is not a geodesic (and does not correspond to a closed geodesic in any obvious way). (However, observe that γ can be represented by a strongly stationary 1-cycle from Γ_3.)

So, the stationarity of γ does not guarantee the smoothness of curves γ_i, which can have many points where they fail to be smooth and, thus, consist of many geodesic segments. (Moreover, the number of non-smooth points of γ_i can even be infinite.) However, if γ is stationary and γ_i is C^2-smooth, then the formula for the first variation of the arclength immediately implies that γ_i is a geodesic (after a suitable reparametrization). So, in the presence of condition 1) the condition 2) is essentially equivalent (up to a reparametrization) to the smoothness of γ_i. Thus, the purpose of condition 2) is to ensure that k is the number of geodesic segments forming γ.

Vice versa, assume that the condition 2) is satisfied. Then the formula for the first variation of the arclength implies that the condition 1) becomes equivalent to the following condition: Assume that several of the curves γ_i share a common endpoint p. Consider all non-constant curves γ_i such that p is one of their endpoints. For each of these curves consider the unit tangent vector at p directed from p. Then the sum of these vectors must be zero. So, we see that the Definition 3 is equivalent to Definition 1.

The stationarity condition 1) comes from geometric measure theory, where it is used to define stationary varifolds (cf. [A2], [P]). Note that condition 2) implies that a strongly stationary 1-cycle $\gamma \in \Gamma_k$ has an additional degree of stationarity: The cycle γ is stationary with respect to variations of γ associated with vector fields along curves γ_i such that the values of these vector fields at common endpoints of γ_i and γ_j (for all i, j) are equal. (The last requirement implies, in particular, that if γ_i is a geodesic loop, then the values of the corresponding vector field at the beginning and the end of γ_i must be the same.) Note that the main difference between a vector field along γ_i and a restriction of a vector field on the ambient manifold to γ_i is that the former can have different values for values of the parameter corresponding to a point of self-intersection of γ_i. It is because of this additional degree of stationarity we decided to adopt the term "strongly stationary" in Definition 3.

Note that a strongly stationary 1-cycle made of one geodesic segment must be a geodesic loop and therefore must be a closed geodesic. (Two unit vectors tangent to the geodesic loop at its origin must cancel each other.) A strongly stationary 1-cycle made of two geodesic segments either consists of two geodesic segments connecting two different points or consists of two geodesic loops. In the first case it is easy to see that it is a closed geodesic. In the second case we have two subcases. If these geodesic loops are based at different points, then both of them must be closed geodesics. If they are based at the same origin, then we see that the sum of the four unit tangent vectors at the origin of the loops must be equal to zero (see Fig. 2(a)). If the dimension of the manifold is greater than two, then, in principle, these two geodesic loops need not form a closed geodesic. However, if the dimension of the manifold is equal to 2, then this condition implies that the strongly stationary 1-cycle is just a self-intersecting closed geodesic. So, we obtain the following lemma.

Lemma. *A non-trivial strongly stationary 1-cycle made of* 2 *segments on a two-dimensional manifold is either a closed geodesic or the union of a closed geodesic and a point or a closed geodesic.*

Remark. The assertion of this Lemma (with a minor inaccuracy) can be found in [ClCo], p. 547-548.

3. A Morse-theoretic type lemma for Γ_k

The main technical results of this section resemble Theorem 4.3 in [P] (though they do not directly follow from it). They also resemble a basic result from the Morse theory asserting that if there are no critical points of a smooth function $F: M \longrightarrow R$ on a compact manifold M in the set $F^{-1}([x_1, x_2])$ then the sublevel set $F^{-1}((-\infty, x_1])$ is a deformation retract of the sublevel set $F^{-1}((-\infty, x_2])$. (The deformation retraction can be obtained using the gradient flow of F.) Our goal is to obtain a result of such type for the length functional on Γ_k. The main technical problem is that Γ_k is not an infinite-dimensional manifold, but consists of finitely many intersecting pieces (each of which is an infinite-dimensional manifold).

In this section we are going to prove that in the absence of strongly stationary 1-cycles, spheres in Γ_k^x are contractible to a point, under a certain additional condition. This is done in two steps. The first step is to show that if there is no strongly stationary 1-cycles of length $\leq x$ then Γ_k^x can be deformed to the zero level, (see Lemma 3). In particular, that implies that spheres can be deformed to the zero level. The second step is to contract the spheres in the zero level to a point, (see Proposition 4).

Lemma 3. *Assume that there are no non-trivial strongly stationary 1-cycles on* M^n *of length* $\leq x$ *made of* k *geodesic segments. Then* Γ_k^0 *is a deformation retract of* Γ_k^x.

Proposition 4. *Assume that there are no non-trivial strongly stationary 1-cycles on* M^n *of length* $\leq x$ *made of* k *geodesic segments. Let* i *be an arbitrary positive integer number, and* $F: S^i \longrightarrow \Gamma_k^x$ *be a continuous map. For each* $j = 1, \ldots, k$ *consider the map* $F_j: S^i \longrightarrow M^n$ *defined by the formula* $F_j(p) = (F(p))_j(0.5)$ *for any* $p \in S^i$. *If all these maps* F_j *are contractible, then* F *is contractible.*

Remark. Of course, the choice of the point $0.5 \in [0,1]$ is completely arbitrary.

We could choose instead, for example, one of the endpoints of the interval $[0,1]$.

Proof of Proposition 4 assuming Lemma 3: Lemma 3 implies that there exists a homotopy H between F and a map of S^i into Γ_k^0. Note that $H(1)$ can be regarded as a map of S^i into $(M^n)^k$. Therefore it is sufficient to check only that k maps $H(1)_j$ are contractible. But each of these maps can be connected with F_j by a homotopy Q that can be defined by the formula $Q(t)(p) = (H(t)(p))_j(0.5)$ for any $t \in [0,1]$, $p \in S^i$. QED.

Proof of Lemma 3:

The proof of Lemma 3 is long but not difficult. Here we present a short version of the proof.

We will present a detailed proof of this Lemma in Appendix A to this paper.

3.1 Reduction to a finite-dimensional case A standard and very old idea in the study of closed geodesics (apparently due to Birkhoff) is to use a length non-increasing deformation of the space of all piecewise-smooth closed curves on a Riemannian manifold to its finite-dimensional subspace that consists of piecewise geodesics.

We can apply this idea in our situation. It is easy to see that there exists a deformation of Γ_k^x to a finite-dimensional space $g_{k,N}^x$ defined as the space of all elements of Γ_k^x such that each of its k segments is a broken geodesic with N segments of length $\leq \mathrm{inj}(M^n)/4$, (see fig. 5). Here $\mathrm{inj}(M^n)$ denotes the injectivity radius of M^n and $N = N(M^n, x)$ is an explicit large number. (For example, one can take $N = [4x/\mathrm{inj}(M^n)]+1$.) We call this deformation *the Birkhoff deformation.*

Note that this deformation is not the Birkhoff curve-shortening process used in many papers about closed geodesics but just its first stage. Also, note that the endpoints of k segments remain fixed during this deformation.

3.2 For our purposes we need the spaces $G_{k,N}^x$ defined almost as $g_{k,N}^x$ with only one distinction: Each of small geodesic segments is allowed to have length $\leq \mathrm{inj}(M^n)/2$ instead of $\mathrm{inj}(M^n)/4$ in the definition of $g_{k,N}^x$. Now our goal will be to prove that there exists a deformation of $g_{k,N}^x$ into $g_{k,N}^0$ inside $G_{k,N}^x$ such that points of $g_{k,N}^0$ are fixed. This assertion immediately implies Lemma 3.

3.3 The vector of the steepest descent The idea is to construct a flow that behaves like a gradient flow. It is not difficult to see what is the direction of the steepest descent for the length functional on $G_{k,N}^x$: A tangent vector to an element of $G^x_{k,N}$ can be thought as a collection of tangent vectors to M^n at every endpoint of many geodesic segments that compose the element. For the steepest descent vector each of these tangent vectors is equal to the sum of unit vectors tangent to all geodesic segments meeting at the endpoint and directed from the endpoint, (see fig. 6). However the steepest descent vector is not a continuous function on $G^x_{k,N}$, (see fig. 7).

3.4 Therefore choose a fine net in $G^x_{k,N}$, an open covering of $G^x_{k,N}$ by small metric balls centered at the points of the net and obtain the desired gradient-like vector field as a linear combination of the steepest descent vectors at the points of the net with coefficients equal to functions of a partition of unity subordinate to the open covering. Here we use the parallel translation along geodesics on M^n in order to translate a tangent vector at a point of M^n to all sufficiently close points. (Thus, a steepest descent vector at a point of $G^x_{k,N}$ can be translated to all sufficiently close points.)

3.5 A small technical complication arises due to the fact that the steepest descent vectors are defined only at the points of $G^x_{k,N} \setminus G^0_{k,N}$, and this space is not compact. Therefore, if we would like to go all the way down to $G^0_{k,N}$ (and not just to $G^\delta_{k,N}$ for some small $\delta > 0$) we need to consider infinite countable nets on $G^x_{k,N} \setminus G^0_{k,N}$ and locally finite open coverings centered at the points of the net on the previous step. As the result a priori we do not have a uniform positive lower bound for the speed of change of the length, when the length approaches 0.

Yet one can derive such a uniform bound by observing that each element of $G^\delta_{k,N}$ for a small δ consists of several connected components that are located in very small open balls in M^n that are very close to the balls of the same radius in $\mathbb{R}^n$. Therefore the speed of change of the length is very close to the speed of the change of the length of corresponding small parametrized 1-cycles in $\mathbb{R}^n$. But in the Euclidean space the steepest descent vectors and the norm of the gradient of the length functional are scale invariant. Hence we can rescale a connected 1-cycle in $\mathbb{R}^n$ so that the maximal length of one of its segments is equal to one without changing the speed of decrease of the length under the flow. Now an obvious compactness argument yields a desired uniform positive lower bound for the speed of change of the length. (We need such a bound in order to be sure that $G^0_{k,N}$ will be reached in a finite time.)

3.6 Another technical difficulty can arise when the value of the length of the considered 1-cycles is $\geq \mathrm{inj}(M^n)/4$. Namely, note that, whereas the length of the element of $G^x_{k,N}$ decreases under the constructed flow, the length of its individual segments can increase. Since we move the endpoints of the segments and connect them by the shortest geodesics, we do not want the distance between two points that are supposed to be connected by a geodesic to become $\geq \mathrm{inj}(M^n)/2$. We also need this restriction because we would like to stay inside $G^x_{k,N}$ during the deformation.

In order to avoid this problem we can proceed as follows. We follow the flow for a small fixed value of time t_0 (for example, one can take $t_0 = \mathrm{inj}(M^n)/(8k(N+1))$). Then we apply the Birkhoff deformation that makes the lengths of all small geodesic segments to be smaller than $\mathrm{inj}(M^n)/4$. Then we follow the flow for time $t = t_0$, then apply the Birkhoff deformation, etc. It is clear that we need to stop and perform the Birkhoff deformation only finitely many times before we reach $g^{inj(M^n)/4}_{k,N}$. (After we reach $g^{inj(M^n)/4}_{k,N}$ this problem cannot occur anymore, and we do not need to apply the Birkhoff deformation.) QED

3.7 A stronger version of Lemma 3. Subsections 3.1-3.6 contain an outline of the proof of Lemma 3. Now we are going to explain how one can prove a slightly stronger version of Lemma 3 (and therefore of Proposition 4). (Again the complete details can be found in Appendix A.) Namely, we can ensure that *the type* of elements of Γ^x_k and of $G^x_{k,N}$ does not change during the deformation, where the type is defined as follows:

3.8 Types of elements of Γ^x_k. We define types of elements of Γ_k as equivalence classes. Two elements are equivalent if 1) The sets of segments that are constant curves are the same; 2) The endpoints of k segments merge with each other in exactly the same fashion. (Note that all segments are numbered.) We refer the reader to Appendix A for a more detailed and formal definition of types.

For example, consider types of elements of Γ_2. There are three types where both segments are non-constant, namely, (1) Two (non-trivial) closed curves; (2) One closed curve, where the endpoints of 2 segments are two different points on the curve; and (3) A figure eight curve (all four endpoints of 2 segments merge). There are two types where both segments are constant: (4) A curve that looks as two distinct points; (5) A curve that looks as a point. There are 4 types, when one of two segments is constant and the other is not: (6_i) The i th segment is constant, and it coincides with both endpoints of $(3-i)$ th segment (which, therefore, forms a closed curve); (7_i) The i th segment is constant, and different from the endpoints of the $(3-i)$ th segment; $i = 1, 2$. (In the case 7_i the element looks as a non-trivial closed curve and a point.)

Note that there is a natural partial order on the set of types: We say that a type A is higher than B if an element of type A can be obtained from an element of type B by 1) collapsing a segment to a point without the merging of the two endpoints, (see fig. 8 (a), (b)); 2) merging two endpoints of k segments, (see fig 8 (c)); 3) A finite sequence of operations of the type 1) and 2). For example, for Γ_2 there are the following inequalities between types: $5 > 4 > 7_i > 1$, $3 > 2$, $5 > 6_i > 3 > 1$. On the other hand, 4 and 3 are incomparable. 1 and 2 are also incomparable.

3.9 Preserving the type during deformation. In order to keep the type constant during the deformation we must keep endpoints of segments away from merging. It can be achieved is the following simple fashion. Stratify $G^x_{k,N}$ by strata corresponding to the types. We start the construction of the net and the open covering from the highest type and than proceed to lower and lower types. By doing this we can ensure that a small neighbourhood of a higher type stratum is covered *only* by balls centered at points in this stratum and strata corresponding to even higher types and does not intersect balls centered at the points of the net located in lower type strata. We are going to consider such a covering of $G^x_{k,N} \setminus G^0_{k,N}$ instead of an arbitrary covering used in 3.4 above. Then proceed as in 3.4 to construct a flow on $G^x_{k,N} \setminus G^0_{k,N}$.

Lemma. *The type of each element of* $G^x_{k,N} \setminus G^0_{k,N}$ *remains constant under this flow.*

Sketch of the proof: The change of type of a parametrized 1-cycle is possible only when two its endpoints merge. But before colliding they need to become close to each other. When two endpoints in a parametrized 1-cycle become very close to merging, they will start moving along trajectories of the same flow on M^n . To explain this assertion assume for simplicity that the considered element of $G^x_{k,N}$ is covered by exactly one metric ball centered at a point in a higher type stratum where these two endpoints have already merged into one point m . (The general case can be treated in a similar fashion; see Appendix A.) Then by the virtue of the construction explained in 3.4 these two endpoints will be moving along the trajectories of a vector field obtained from a tangent vector to M^n at m by the parallel translation along the unique shortest geodesics from m to these two points. The two endpoints will be moving along the trajectories of this flow on M^n either until they will become sufficiently far apart at a later time, or until the flow reaches $G^0_{k,N}$. But the uniqueness theorem for ODE implies that points moving along the trajectories of the same flow on M^n cannot collide. (And recall that during Birkhoff deformation stages endpoints of the intervals remail fixed.) Therefore the type of the parametrized 1-cycle cannot change. QED.

Note that keeping the type constant does not really give us much. Informally speaking, we keep points from merging in a somewhat artificial way, and, anyway, move them almost as if they have already been merged into one point. Yet this feature of our flow turns out to be a (non-essential) convenience when we build singular chains out of the discs in the spaces of parametrized 1-cycles as explained in the next section. (More presisely, as the result, we will be able to use a very transparent version of Almgren correspondance explained in the next section. An alternative is to use a stronger version of Almgren correspondance outlined in Appendix B. In this case one does not need the strenghening of Lemma 3 proved in subsections 3.7-3.9 which then can be omitted.)

Observe also that the assertion of Proposition 4 will hold with the same proof for maps $F : S^i \longrightarrow Z^x_{(k)}$. For these maps there will be no need to check if the maps F_j are contractible since $Z^0_{(k)}$ consists of one point, namely the zero cycle.

Finally observe that if M^n is diffeomorphic to S^2 we can combine Lemma 3 in the case of $k = 2$ with the following observation (stated as Lemma in section 2.2 above): Each non-trivial strongly stationary 1-cycle made of 2 geodesic segments on a two-dimensional manifold is either a closed geodesic or union of a closed geodesic with a point or another closed geodesic. As the result we obtain an elementary proof of the following assertion used in our paper [NR]. (This assertion first appeared in [ClCo].)

Proposition 5. *Let M be a Riemannian manifold diffeomorphic to S^2. Let Γ^x_2 denote the space of parametrized 1-cycles on M made of 2 segments. Assume that for some x there exists a non-contractible map $f : S^1 \longrightarrow \Gamma^x_2$. Then there exists* *a non-trivial closed geodesic of length* $\leq x$ *on* M *.*

Proof. Proposition 4 implies the existence of a non-trivial strongly stationary 1-cycle in Γ_2^x. Since M is two-dimensional, Lemma in section 2.2 implies that this 1-cycle is either a closed geodesic or contains a closed geodesic (of smaller length) as one of its two connected components. QED.

4. Almgren correspondence

In [A] F. Almgren proved a general theorem that, in particular, immediately implies that for every m the groups $\pi_m(Z_1(M^n, Z))$ and $H_{m+1}(M^n; Z)$ are isomorphic.

Here we are going to adapt and simplify the Almgren construction, (see [A]) to the spaces Γ that consist of parametrized 1-cycles made of finitely many closed curves.

Assume that we are given a continuous map A from a compact polyhedron $|K|$ into Γ_k. We would like to assign to A a $(dim|K|+1)$ - dimensional singular chain on M^n. This assignment will be defined for all A that have a certain property that we will call a local triviality, see def. 5 below.

The assignment will have the following property: if A is a map of $S^m = \partial D^{m+1}$ that is the restriction of a locally trivial map B of D^{m+1} to ∂D^{m+1}, then the chain assigned to A is the boundary of the chain assigned to B. This assignement will be called *the Almgren correspondance.*

Consider the space Γ_k. It can be regarded as a subset of the topological space of all maps of the disjoint union of k copies of $[0,1]$ into M^n. Therefore we can assign a continuous map θ from $X = |K| \times \bigcup_{i=1}^{k}[0,1]_i$ into M^n to any given map $A : |K| \longrightarrow \Gamma_k^x$ in the standard way: For each $x \in |K|$, $i \in \{1, \ldots, k\}$, $t \in [0,1]_i$ $\theta(x,i,t) = A(x)_i(t)$. Further, since elements of Γ_k are parametrized cycles we can identify points of $2k$ sets $|K| \times \{0\}$, $|K| \times \{1\}$ that are mapped into the same points of M^n. Denote the resulting quotient of X by X_A. The map θ factors through X_A. Denote the resulting map of X_A to M^n by θ_A. Note that the quotient X_A can be quite complicated.

However, for our purposes we will only need the situation when X_A can be triangulated with a finite number of simplices. Moreover, we are going to make an even stronger assumption that will always hold when we apply the Almgren correspondance.

Definition 5. *A map* $A : |K| \longrightarrow \Gamma_k$ *is called locally trivial if* $|K|$ *admits a simplicial subdivision with the following property: For any open simplex* σ *of any dimension of this subdivision all parametrized 1-cycles* $A(t) \in \Gamma_k$, $t \in \sigma$ *have the same type.*

Now we can triangulate X_A so that the map $\pi : X_A \longrightarrow |K|$ induced by the projection of X on $|K|$ becomes a simplicial map such that the inverse image of each open simplex σ of $|K|$ under π is the product, and the restriction of π to $\pi^{-1}(\sigma)$ is the projection. We will also call such triangulations of X_A *locally trivial.*

Consider the singular chain in M^n corresponding to the simplicial map θ_A from X_A endowed with a simplicial triangulation. The local triviality makes the following assertions evident: If $|K| = S^{m-1}$ then the resulting singular m-chain will be a singular cycle. Its homology class does not depend on the choice of a locally trivial triangulation of X_A. If A is a map of S^{m-1} to Γ_k obtained as the restriction of a locally trivial map $B : D^m \longrightarrow \Gamma_k$ to ∂D^m, then for any locally trivial triangulation of X_B the boundary of the corresponding singular $(m+1)$-chain in M^n will be the singular chain obtained from $X_A \subset X_B$. Therefore the singular m-cycle in M^n assigned to $A : S^{m-1} \longrightarrow M^n$ will represent 0 in $H_m(M^n)$ if A is contractible.

Note that if f is a map of S^{i-1} into Γ_k^x satisfying the local triviality assumption, and $H : S^{i-1} \times [0,1] \longrightarrow \Gamma_k^x$ is the homotopy between f and a map g of S^{i-1} into Γ_k^0 constructed as in the proof of Lemma 3, (in the situation when there are no non-trivial strongly stationary parametrized 1-cycles in Γ_k^x), then H also satisfies this assumption. This assertion immediately follows from the fact that the type does not change during the deformation of Γ_k^x into Γ_k^0 constructed during the proof of Lemma 3, (see section 3.7 - 3.9 or Appendix A for more details.) This observation will imply the local triviality of A in situations that arise in the course of proving Theorems 1 and 2 in Section 5.

For the sake of completeness we are also going to sketch how to modify the above version of the Almgren correspondence to make it work in the general case, when we do not even have the triangulability of X_A. (This construction will not be used in the present paper.) The reader will find this construction in Appendix B to this paper.

5. Proofs of Theorems 1 and 2.

Proof of Theorem 1. A. Outline of the proof. Assume that $\alpha(M^n)$ and, in particular, $l(M^n)$ is greater than $\frac{(n+2)!d}{3}$. Thus, each closed curve of length $\frac{(n+2)!d}{3}$ can be contracted to a point by a homotopy that does not increase its length.

Consider a map $f : S^q \longrightarrow M^n$ representing a non-zero element of $\pi_q(M^n)$, where S^q is the standard sphere with a fine triangulation. Let $[S^q]$ be the fundamental class of S^q. Since the map is non-contractible, $f_*([S^q]) \neq 0 \in H_q(M^n)$. Let D^{q+1} be a disc that has S^q as its boundary. Triangulate D^{q+1} as the cone over the triangulation of S^q (introducing one new 0-dimensional simplex at the centre of D^{q+1}). We will try to construct a singular $(q+1)$-chain in M^n, such that $f_*[S^n]$ will be its boundary, which is clearly impossible and will result in a desired contradiction.

We are going to proceed inductively assigning an i-dimensional singular chain in M^n to each i-dimensional simplex of D^{q+1} on the i-th step. The induction starts from assigning an arbitrary point of M^n to the center of D^{q+1}. The new 1-dimensional simplices will correspond to the shortest geodesics connecting the images of the endpoints in M^n. (These geodesics are regarded as singular simplices in M^n here.) Each new 2-dimensional simplex $\sigma \subset D^{q+1} \setminus S^q$ will correspond to a signular chain that consists of one singular 2-simplex. This simplex will be provided by a length non-increasing homotopy that contracts the image of the boundary of σ in M^n. (Such length-nonincreasing homotopies exist because of our assumption.)

Now consider higher dimensions. The boundary of the singular chain that corresponds to an arbitrary simplex σ^i will be equal to the signed sum of chains assigned to simplices of the boundary of σ^i. These signs will be the same as the signs with which the corresponding simplices enter $\partial\sigma^i$. These singular i-chains will be obtained from $(i-1)$-dimensional discs in the space of parametrized 1-cycles, particularly in $\Gamma_{k(i-1)}$ for some function $k(i)$. Here we will use the Almgren correspondence between discs and chains explained in section 4. In turn, these $(i-1)$-dimensional discs are obtained by contracting $(i-2)$-dimensional spheres in $\Gamma_{k(i-2)}$. And these $(i-2)$-dimensional spheres are constructed from

$(i-2)$-dimensional discs in $\Gamma_{k(i-2)}$ corresponding to simplices of $\partial\sigma^i$ that were constructed on the previous stage of induction. (The construction of $(i-2)$-dimensional spheres from the collection of $(i-2)$-dimensional discs will be explained below.)

Alternatively we can describe the same procedure with more details and from a somewhat different perspective in the following way: After the first three steps of the induction process corresponding to simplices of dimension 0, 1 and 2 in D^{q+1} we obtain a collection of maps of $D^1 \longrightarrow \Gamma_{k(1)} \longrightarrow Z_{(k(1))}$, where $k(1)=3$. These maps correspond to contractions of boundaries of 2-simplices in the image in M^n of the triangles in $D^{q+1}\setminus S^q$ that do not increase the length. Then, inductively, for each simplex σ^i in the considered triangulation of D^{q+1}, ($i=3,4,\dots,q+1$), we do the following:

1) Construct a map of S^{i-2} into $Z_{(k(i-1))}$ using $(i+1)$ maps of D^{i-2} into $Z_{(k(i-2))}$ corresponding to $i+1$ $(i-1)$-dimensional simplices in the boundary of σ^i and obtained on the previous step of induction. In order to do that we first observe that $(Z_{(k(i-2))})^{i+1}\subset Z_{((i+1)k(i-1))}$. Therefore we obtain a map ω of $D^{i-2}\longrightarrow Z_{(k(i-1)}$, where, by definition, $k(i-1)=(i+1)k(i-2)$. The restriction of this map to ∂D^{i-2} sends each point to the same point of $Z_{(k(i-1))}$, namely the zero cycle. This happens because the sum of boundaries of $(i+1)$ oriented $(i-1)$-dimensional simplices constituting the boundary of σ^i is zero as a chain. In other words, each simplex of dimension $(i-2)$ appears in this sum the same number of times with each of two opposite orientations. This leads to the cancellation of all 1-dimensional (non-parametrized!) cycles in the corresponding sum. Therefore ω factors through S^{i-1}: We just map the boundary of D^{i-1} to a point that will be mapped to the zero cycle.

2) We use our assumption about non-existence of sufficiently short non-trivial strongly stationary 1-cycles and Proposition 4 to obtain a map of D^{i-1} into $Z_{(k(i-1))}$ contracting this map of S^{i-2}. This map of D^{i-1} will correspond to σ^i. (Formally speaking, Proposition 4 involves parametrized cycles instead of non-parametrized cycles. But, as we observed before, Proposition 4 holds and is even easier for non-parametrized cycles. In particular, there is no need to check the contractibility of F_j, since all of them are constant maps into the zero cycle.)

All these maps from discs and spheres into $Z_{(k(i))}$, $i=1,2,\dots,q$, can be lifted to $\Gamma_{k(i)}$. In other words, we can carry out this construction with parametrized cycles $\Gamma_{k(i)}$ instead of $Z_{(k(i))}$. Ad hoc, there will be two extra difficulties here: Firstly, on step 2) we will need to check the contractibility of maps F_j in order to apply Proposition 4. Secondly, the map ω defined during our discussion of step 1) will not map ∂D^{i-2} into the zero cycle anymore. Instead each point of ∂D^{i-2} is mapped into a parametrized 1-cycle that consists from a finite number of segments, so that each segment enter the same number of times with each of two orientations. So, we can pair these segments with opposite orientations and contract them to the point in the middle (which is a parametrized 1-cycle of zero length). As the result, we obtain a homotopy H of $\omega|_{\partial D^{i-2}}$ to a map τ of ∂D^{i-2} into $\Gamma^0_{k(i-1)}$, which can be also regarded as a finite collection of maps from $S^{i-3} = \partial D^{i-2}$ to M^n. If we manage to demonstrate that these maps are null-homotopic, then we can obtain the required map of S^{i-2} into $\Gamma_{k(i-1)}$ by attaching to ω H and a homotopy G contracting τ. (Here ω can be regarded as a map of a lower hemisphere of S^{i-2}, G can be regarded as a map of an upper hemisphere, and H can be regarded as a map of the spherical annulus between the hemispheres.) So, all our technical difficulties can be reduced to verification that certain maps of spheres into M^n are null-homotopic.

This problem can be dealt with in two ways. First, we can just observe that by our notations all homotopy groups of M^n in the considered dimensions are trivial (because these dimensions are less than q), so the problem disappears. Alternatively, we can examine the geometry of these maps. It turns out that each of this maps looks like a composition of a flattening of the sphere into the double disc of the same dimension and a map of the disc into M^n, so it is null-homotopic in the obvious way.

We continue this inductive procedure until i becomes equal to $q+1$. As the result we obtain maps of D^q into $\Gamma_{k(q)}$ corresponding to every $(q+1)$-dimensional simplex of the triangulation of D^{q+1}.

Then we will apply the Almgren correspondence to each of the resulting maps of D^q into $\Gamma_{k(q)}$ and sum the resulting singular $(q+1)$-chains in M^n. The result will be the required $(q+1)$-chain that fills the singular cycle $f_*([S^q])$, and we obtain the desired contradiction.

B. Details. We will begin with the 0**-skeleton** of $D^{q+1} \setminus S^q$ that consists of the point p, the center of the disc. We will assign to p a singular 0-chain that corresponds to an arbitrary point $\tilde{p} \in M^n$. Now we will proceed to the 1**-skeleton**: we will assign to the 1-simplices of the form $[v_i, p]$ the singular 1-chains that correspond to minimal geodesics in M^n that connect $\tilde{p}$ and $\tilde{v}_i = f(v_i)$ Next, we consider the 2**-skeleton**: Let $\sigma^2 = [v_i, v_j, p]$ be a 2-simplex of $D^{q+1} \setminus S^q$. Consider its boundary $\partial\sigma^2$ and the corresponding singular 1-chain on M^n, which equals to $[\tilde{v}_j, \tilde{p}] - [\tilde{v}_i, \tilde{p}] + [\tilde{v}_i, \tilde{v}_j]$. This can be viewed as a curve of length $\leq 2d + \epsilon$. By our assumption, there is no closed geodesics of length smaller than or equal to $2d + \epsilon$, so there is a curve shortening homotopy that connects this curve with a point. Therefore, we assign to this 2-simplex a singular 2-chain consisting of one singular 2-simplex that corresponds the surface generated by this homotopy. The "extension" to the 3**-skeleton** will be somewhat different. Let $\sigma^3 = [v_{i_0}, v_{i_1}, v_{i_2}, v_{i_3}]$ be a 3-simplex of $D^{q+1} \setminus S^q$. We want to find a singular 3-chain to assign to this simplex. Consider $\partial\sigma^3$. There is a singular 2-chain assigned to the boundary of this simplex, which can also be viewed as a 2-sphere in M^n of a particular shape. Namely, to each of the faces of the boundary not in S^q there was assigned a surface generated by a curve shortening homotopy. Without any loss of generality we can assume that the chosen fine triangulation of S^q and the map of S^q into M^n were chosen so that any two-dimensional simplex of the triangulation S^q is also mapped into the surface obtained by contracting its boundary in M^n by a homotopy that does not increase the length. As we will see, this 2-sphere corresponds to a 1-sphere in $\Gamma_{12}^{6d+6\epsilon}$. (See figure 1 to understand how this 1-sphere is constructed.) In order to describe this correspondence let $e_1 = [\tilde{v}_{i_0}, \tilde{v}_{i_1}], e_2 = [\tilde{v}_{i_0}, \tilde{v}_{i_2}], e_3 = [\tilde{v}_{i_0}, \tilde{v}_{i_3}], e_4 = [\tilde{v}_{i_1}, \tilde{v}_{i_2}], e_5 = [\tilde{v}_{i_1}, \tilde{v}_{i_3}], e_6 = [\tilde{v}_{i_2}, \tilde{v}_{i_3}]$, where each $[\tilde{v}_{i_s}, \tilde{v}_{i_t}]$ is a minimal geodesic segment on the manifold. Then we will let $\gamma_1 = e_1 + e_5 - e_3, \gamma_2 = -e_1 + e_2 - e_4, \gamma_3 = -e_2 + e_3 - e_6, \gamma_4 = e_6 - e_5 + e_4$. Let x_i be a point to which γ_i contracts for $i = 1, ..., 4$. Then the 1-sphere in the space of 1-cycles will be constructed as follows: let $\tilde{f}_i : D^2 \longrightarrow M^n, i = 1, ..., 4$ be each of the four discs that make the 2-sphere in M^n. Those discs correspond to four maps $f_i : [0,1] \longrightarrow Z_1(M^n, Z)$, such that $f_i(0) = T_{\{x_i\}} = 0, f_i(1) = T_{\gamma_i}$. (Recall that T_A denotes a linear functional on the space of 1-forms of the manifold defined by the formula $T_A(\lambda) = \int_A \lambda$.) These maps are precisely curve-shortening homotopies used to obtain $\tilde{f}_i$; for any $t \in [0,1]$ $f_i(t)$ is a 1-cycle that consists of one closed curve. It can be regarded as an element of $Z_{(3)}$ if we represent γ_i as the collection of three curves (=three sides of the triangle) glued at their endpoints, and will keep track of these three curves during homotopies contracting γ_i. Now we will let $G_1 : [0,1] \longrightarrow Z_{(12)}$ be the map that for each $q \in [0,1]$ assigns $\Sigma_{i=1}^4 T_{f_i(q)}$, (see figure 1(b)). Note that $G_1(0) = \Sigma_{i=0}^4 T_{\{x_i\}}$, which is the zero cycle, (see figure 1(a)) and that $G_1(1) = \Sigma_{i=0}^4 T_{\gamma_i}$, which is also the zero cycle, (see figure 1(c)). Thus, we obtain a map from S^1 to $Z_1(M^n, \mathbb{Z})$.

In order to obtain the corresponding map of S^1 into Γ_{12} note that the restriction of G_1 to $[0,1)$ lifts to Γ_{12} in the obvious way. It remains to exhibit a homotopy between $\bigcup_{i=1}^4 \gamma_i$ to $\bigcup_{i=1}^4 \{x_i\}$, where each $\{x_i\}$ is counted three times and is regarded as a constant segment, in Γ_{12} that lies over the zero cycle in $Z_{(12)}$ in order to close the circle. This can be achieved by first cancelling in a continuous way six pairs of edges e_i with the opposite orientations to a point, which is obviously possible (each pair is connected over itself to the point corresponding to $t = \frac{1}{2}$ counted twice), and then connecting 12-tuples of these points regarded as an element of Γ_{12} with the constant 1-cycle $\{x_1, x_2, x_3, x_4\}$ regarded as the cycle from Γ_{12} (each point is counted three times) using twelve continuous paths. These paths follow our homotopies restricted to the points of edges of γ_i corresponding to $t = 0.5$ in the chosen parametrization of these edges. As the result we obtain a lifting to Γ_{12} of a map that differs from G_1 only by a reparametrization.

Further, for sufficiently small $\epsilon > 0$ Proposition 4 implies that the lifting of the map $S^1 \longrightarrow Z_{(12)}^{6d+6\epsilon}$ to Γ_{12} is also contractible. Indeed, we just need to verify the contractibility of 12 maps of $S^1 \longrightarrow M^n$. (These maps were denoted by F_j in the text of Proposition 4.) Of course, this fact follows from the simply-connectedness on M^n. However, there is even a more straightforward geometric reason for contractibility of these 12 circles in M^n: each of them is formed by the trajectory of a homotopy $\tilde{f}_i$ from x_i to a point in the middle of a geodesic segment e_j traversed two times in the opposite directions.

Observe that using the Almgren correspondence we see that this disc corresponds to a 3-chain that we will denote $\tilde{C}_{v_{i_0},\dots,v_{i_3}}$ in M^n. It immediately follows from our construction that the boundary of this chain is the (signed) sum of singular chains (=simplices) assigned to 4 2-dimensional simplices in the boundary of the considered 3-dimensional simplex of $D^{q+1} \setminus S^q$. So we will assign $\tilde{C}_{v_{i_0},\dots,v_{i_3}}$ to the simplex σ^3.

Now consider the extension to the 4**-skeleton** of $D^{q+1} \setminus S^q$. (see Fig. 4.) Consider any 4-simplex of $D^{q+1} \setminus \partial D^{q+1}$ $\sigma^4 = [v_{i_0}, ..., v_{i_4}]$. The 3-dimensional cycle in M^n $C_{v_{i_0},...,v_{i_4}} = \sum_{j=0}^4 (-1)^j \tilde{C}_{v_{i_0},...,\hat{v}_{i_j},...,v_{i_4}}$ corresponds to the boundary $\partial\sigma^4$ of this simplex. First, we construct the corresponding map of the 2-disc to $Z^{5(6d+6\epsilon)}_{(60)} \subset Z_1(M^n, \mathbb{Z})$, that takes the boundary of this disc to the zero cycle. This map, denoted G_2 will be constructed as follows: let $f_j : \bar{D}^2 \longrightarrow Z^{6d+6\epsilon}_{(12)} \subset Z_1(M^n, Z)$ be the map corresponding to $(-1)^j \tilde{C}_{v_{i_0},...,\hat{v}_{i_j},...,v_{i_4}}$ constructed during the previous step of our induction process. Then let $G_2(q) = \Sigma_{j=0}^4 T_{f_j(q)}$ for any $q \in \bar{D}^2$. Observe that this map can be lifted to $\Gamma^{30d+30\epsilon}_{(60)}$ in the obvious way.

Let us now examine the restriction of G_2 to $\partial\bar{D}^2$. (see Fig. 4 (c).) We see that for any $q \in \partial\bar{D}^2, G_2(q)$ corresponds to the union of 10 pairs of closed curves, where each pair will contain the same curve with two different orientations. In other words, the corresponding 1-cycles will have opposite signs, and will cancel. As the result we obtain a zero cycle.

Thus, we obtained a 2-sphere in the space of non-parametrized 1-cycles. We would like to apply Proposition 4. In order to do that we need first to lift G_2 to the map $\tilde{G}_2 : \bar{D}^2 \longrightarrow \Gamma_{60}$ and examine what happens to the boundary of the disc under this map. Each point on the boundary is mapped to the 30 pairs of segments in M^n. Each pair consists of the same segment with opposite orientations. We want to construct a homotopy between $\tilde{G}_2 : \partial\bar{D}^2 \longrightarrow \Gamma_{60}$ and a constant map, i.e. a map that will take a circle to 60 point curves. To construct this homotopy we cancel pairs of parametrized 1-cycles corresponding to the 1-cycles with opposite orientations mentioned in the previous paragraph in a continuous way. We contract each pair $\gamma \bigcup -\gamma$ to $\gamma(0.5)$ over γ. (See Fig. 4 (d).) Thus, we obtain a circle in the space Γ_{60}, where each point $p \in S^1$ corresponds to 60 constant paths (those paths are different for $p \neq p'$). This circle can be interpreted as 60 circles on M^n. Since M^n is simply connected, these circles can be contracted to an arbitrary point in M^n. (There is also a more straightforward reason why these circles are contractible, see Fig. 4(e). This reason will be explained during the proof of Theorem 2 below, where we encounter a similar situation, but M^n cannot be assumed to be simply connected.) After contracting them we can obtain a point in the space Γ_{60} made of 60 constant segments. So, combining $\tilde{G}_2$ with these two homotopies, we obtain a map of the 2-disc into Γ_{60} such that its boundary is mapped into a point composed of 60 constant segments. We can factor this map through the sphere S^2 obtained from the disc by identifying its boundary to a point (say, the north pole of the sphere. In this case the southern hemisphere is mapped by $\tilde{G}_2$, and the northern hemisphere is mapped into the subset of Γ_{60} that corresponds to the zero cycle (i.e. in $I^{-1}(0)$).) So Proposition 4 applies: We have constructed a 2-dimensional sphere in the space Γ_{60}, and can conclude that either this sphere can be contracted along the cycles of mass $\leq 30(d+\epsilon)$, or we have a stationary 1-cycle of mass controlled from above by this bound. (Here the verification of the contractibility of maps F_j defined in the text of Proposition 4 is equivalent to the contractibility of certain 60 2-spheres in M^n. But now we are discussing the case of $q \geq 3$, so M^n is 2-connected.) If ϵ is sufficiently small, then the second case is impossible. In the former case, we obtain a 3-disc in the space of 1-cycles, that corresponds to a 4-chain that we will denote $\tilde{C}_{v_0,\ldots,v_4}$. We will then assign this chain to σ^4.

Now we can continue in the above manner until we fill the original q-dimensional chain $f_*([S^q])$ by a $(q+1)$-dimensional chain in M^n. As a corollary of our assumption nothing will stop us until we construct the desired filling. But as it was said before, this is impossible, and we obtain a contradiction refuting our assumption. The constants $(q+2)!/4$ and $(q+2)!/2$ in the text of Theorem 1 can be explained by the fact that all our 1-cycles consist of at most $4\times 5\times 6\times \ldots \times (q+2)$ closed curves of length not exceeding $2d+\epsilon$, and each of these closed curves consists of three segments. Moreover, a quarter of these closed curves have the length $\leq 3\epsilon$, which tends to 0, as $\epsilon \longrightarrow 0$.

Note that we can get a better estimate when $q=2$. In this case we need to perform the extension process only till the dimension $q+1=3$. We will need "to represent" the union of four maps of D^2 to M^n corresponding to four faces of a 3-dimensional simplex as a map of a circle to Γ_6. (Recall that these four maps where obtained by contracting the maps of boundaries of these discs to a point without increase of their lengths, see Fig. 1). In the body of the proof we mapped a generic point $t \in [0,1]$ into the 1-cycle that corresponds to the union of four curves obtained from homotopies contracting ∂D_i^2 at the moment t (see Fig. 1(b)). In the particular case $q=2$ we can proceed in a slighly different way.

We can start from two points obtained as the result of contraction of the maps of boundaries of D_1^2 and D_2^2 and to pass via cycles made of two closed curves (obtained during the curve-shortening homotopies contracting the maps of ∂D_1^2 and ∂D_2^2) to the cycle made of the images of these two boundaries (see Fig.1). The edge $[v_0, v_2]$ will be passed twice with opposite orientation. Continue the homotopy by cancelling this edge. At the end of this homotopy we obtain the map of the boundary of $D_1^2 \bigcup D_2^2$. Now note that $\partial(D_1^2 \bigcup D_2^2) = \partial(D_3^2 \bigcup D_4^2)$. But we can similarly construct a homotopy between $\partial(D_3^2 \bigcup D_4^2)$ and the zero cycle that uses 1-cycles made of two curves obtained from the curve-shortening homotopies contracting the maps of the boundaries of D_3^2 and D_4^2. Joining these two homotopies we obtain the desired homotopy between the zero 1-cycle and the zero 1-cycle, i.e. the desired circle in the space of 1-cycles that passes through 1-cycles made of not more than two closed curves of length not exceeding $2d+\epsilon$ (each). See the proof of Theorem 1 in [NR] for more details (in the situation when M^n is diffeomorphic to S^2. But this part of the proof is the same there as in the more general situation.) QED.

Proof of Theorem 2. Assume $\alpha(M^n)$ and, in particular, $l(M^n)$ is greater than $(n+1)n^n(n+1)!^{\frac{1}{2}}(n+2)!(volM^n)^{\frac{1}{n}}$. Then $\alpha(M^n)$ (and $l(M^n)$) are greater than $(n+2)!FillRad(M^n)$. The definition of the filling radius implies that M^n bounds in the $(FillRad(M^n)+\delta)$-neighborhood of M^n in $L^\infty(M^n)$. Let W "fill" M^n in the $(FillRad(M^n)+\delta)$-neighborhood of M^n (that is $M^n=\partial W$, if M^n is orientable, and $M^n=\partial W \ (mod\ 2)$, if M^n is not orientable.) Without any loss of generality we can assume that W is a polyhedron.

Suppose W together with M^n is endowed with a very fine triangulation. We are going to try to construct a singular $(n+1)$-chain on M^n such that the boundary of that chain is homologous to the boundary of W (regarded as a chain). That is clearly impossible, so we will obtain a contradiction. We will construct this chain by induction with respect to the dimension of skeleta of W. That is to each i-simplex of W we will assing a singular i-chain on M^n. We will begin with the 0**-skeleton** of W. Let v_i be a vertex of W. Then $F(v_i)=\tilde{v}_i \in M^n=\partial W$, such that $d(v_i,\tilde{v}_i)=d(v_i,M^n)\leq FillRadM^n+\delta$. Suppose $\tilde{v}_i,\tilde{v}_j$ come from the vertices v_i,v_j of some simplex in W. Then $d(\tilde{v}_i,\tilde{v}_j)\leq 2FillRadM^n+3\delta$. (We assume here that the triangulation of W is fine so that the lengths of 1-

simplices of the triangulation are at most δ.) Next, we are going to extend F to the 1**-skeleton**. We will assign to any 1-simplex $[v_i, v_j] \subset W \setminus M^n$ a singular 1-chain that corresponds to a minimal geodesic that connects $\tilde{v}_i$ and $\tilde{v}_j$ of length $\leq 2FillRadM^n + 3\delta$. Now we can see that the boundary of each 2-simplex in W is sent to a singular chain that corresponds to a curve of length $\leq 6FillRadM^n + 9\delta$, (we will assume that all simplices in M^n are already short).

Next we are going to extend to the 2**-skeleton**. Let σ^2 be a 2-simplex of W. Consider its boundary $\partial\sigma^2$ and its corresponding singular 1-chain. There is a curve shortening homotopy that connects the curve corresponding to that chain to a point. So we will map σ^2 to the chain that corresponds to the surface determined by this homotopy. To "extend" F to the 3**-skeleton** of W consider an arbitrary 3-simplex σ^3. Consider its boundary $\partial\sigma^3$ and the corresponding singular 2-chain, which can be viewed as 1-sphere in the space $Z_1(M^n, \mathbb{Z})$ or in Γ_{12} as in the proof of the Theorem 1. This sphere passes through 1-cycles of length $\leq 4(6FillRadM^n + 9\delta)$. Suppose this sphere cannot be contracted via the 1-cycles of smaller mass. Then there exists minimal 1-cycle of length $\leq 4(6FillRadM^n + 9\delta)$ contradicting our assumption. (Here we use Proposition 4 from the previous section. One can check that our spheres in the space of non-parametrized 1-cycles can be lifted to spaces of parametrized 1-cycles exactly as this was done in the proof of Theorem 1 above.) So the above 1-sphere can be "filled" by a disc that passes through 1-cycles of mass not exceeding the above bound. This disc corresponds to a singular 3-chain that has $F(\partial\sigma^3)$ as its boundary. So we will assign this chain to σ^3. The procedure of "extending" to 4-skeleton is similar to the one in the proof of the Theorem 1: At this point "the image" of $\partial\sigma^4$ has been determined and it equals to $C_{v_{i_0},\ldots,v_{i_4}} = (-1)^j \sum_{j=0}^{4} \tilde{C}_{v_{i_0},\ldots,\hat{v}_{i_j},\ldots,v_{i_4}}$. This chain is in fact a 3-dimensional cycle in M^n, and it can be interpreted as a sphere of dim 2 in $Z_1(M^n, \mathbb{Z})$. This sphere is constructed as follows. Let $f_j : \bar{D}^2 \longrightarrow Z_1(M^n, \mathbb{Z})$ be a map that corresponds to $(-1)^j \tilde{C}_{v_{i_0},\ldots,\hat{v}_{i_j},\ldots,v_{i_4}}$. Then, let $G_2 : \bar{D}^2 \longrightarrow Z_1(M^n, \mathbb{Z})$ be a map that assigns to every $q \in \bar{D}^2$ a point $\Sigma_{j=0}^{4} T_{f_j(x)}$. Then it is easy to see that the boundary of the disc is mapped to the zero 1-cycle, and we obtain a 2-sphere in $Z_{(60)}$. Now we want to use Proposition 4, so we need to lift this map of the 2-sphere to Γ_{60}. First we lift G_2 in the obvious way and consider $\tilde{G}_2 : \bar{D}^2 \longrightarrow \Gamma_{60}$. Next consider what happens to $\tilde{G}_2 : \partial\bar{D}^2 \longrightarrow \Gamma_{60}$. We see that each point is mapped to the union of 30 pairs of segments. Each pair consists of the same segment with different orientation. Those segments can be continuously contracted to their middles, (namely contract $\gamma \cup -\gamma$ to $\gamma(0.5)$). Thus we obtain a homotopy between the original circle and the circle that passes through constant parametrized 1-cycles only. This circle corresponds to 60 circles on a manifold, which we want to contract. Now unlike the proof of Theorem 1 above we cannot assume that M^n is simply connected. But we can contract these circles using the following simple construction that will similarly work for all dimensions: Consider the disc $\tilde{G}_2 : \bar{D}^2 \longrightarrow \Gamma_{60}$. For any $p \in \bar{D}^2$ $\tilde{G}_2(p) = \{\gamma_1^p, ..., \gamma_{60}^p\}$. For each p consider $\{\gamma_1^p(0.5), ..., \gamma_{60}^p(0.5)\}$. This determines a 2-dimensional disc in Γ_{60} that passes only through constant parametrized 1-cycles. This disc corresponds to 60 discs on the manifold. Now our circles can all be contracted over these discs, which establishes a homotopy between $\partial \bar{D}^2$ and a point in Γ_{60} . Now we can construct the required map of the 2-dimensional sphere into Γ_{60} providing the desired lifting to Γ_{60} exactly as this was done in the proof of Theorem 1. Also, note that in our present situation maps F_j defined in the text of Proposition 4 are 60 maps of S^2 to M^n defined as follows: For p in the southern hemisphere $F_j(p)$ is defined as $(\tilde{G}_2(p))_j(0.5)$. (Here we identify the southern hemisphere with D^2 .) For p in the northern hemisphere between the equator and a certain parallel $F_j(p)$ is constant on every meridian. (This stage corresponds to contracting oppositely directed pairs of segments to the points in the middle corresponding to $t = 0.5$.) Finally, F_j maps the part of the northern hemisphere north of this parallel using $(\tilde{G}_2(p))_j(0.5)$ again. So, up to a homotopy F_j maps both hemispheres of S^2 in the same way. Hence F_j is contractible by the obvious homotopy. Therefore we can apply Proposition 4: Since our 2-sphere passes only through sufficiently short parametrized 1-cycles and because of our assumption, it can be contracted through sufficiently short parametrized 1-cycles. The 3-disc contracting this sphere corresponds to a 4-chain in M^n filling the 3-cycle we started from. (Here we have used the Almgren correspondence explained in the previous section). And so on.

It becomes obvious, that we can go on like that until we "extend" F to the $(n+1)$-skeleton of W thereby obtaining a $(n+1)$-singular chain in M^n filling the fundamental homology class of M^n which is clearly impossible. (If M^n is non-orientable, then we must reduce the corresponding singular chain modulo $\mathbb{Z}_2$.

By definition, in the non-orientable case the fundamental homology class of M^n is the non-trivial element of $H_n(M^n, \mathbb{Z}_2) = \mathbb{Z}_2$.) The resulting contradiction proves the theorem. QED.

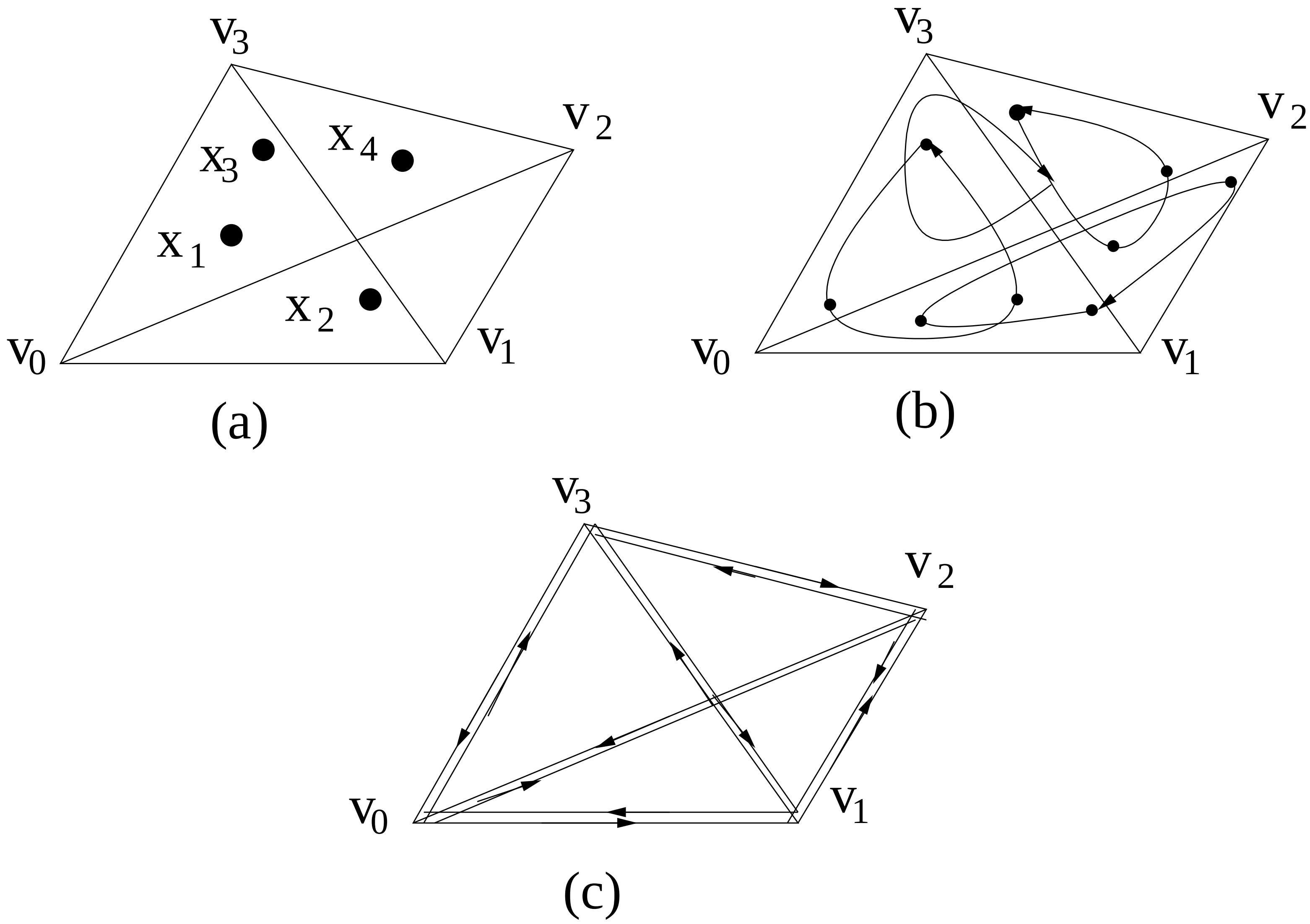


Figure 1.

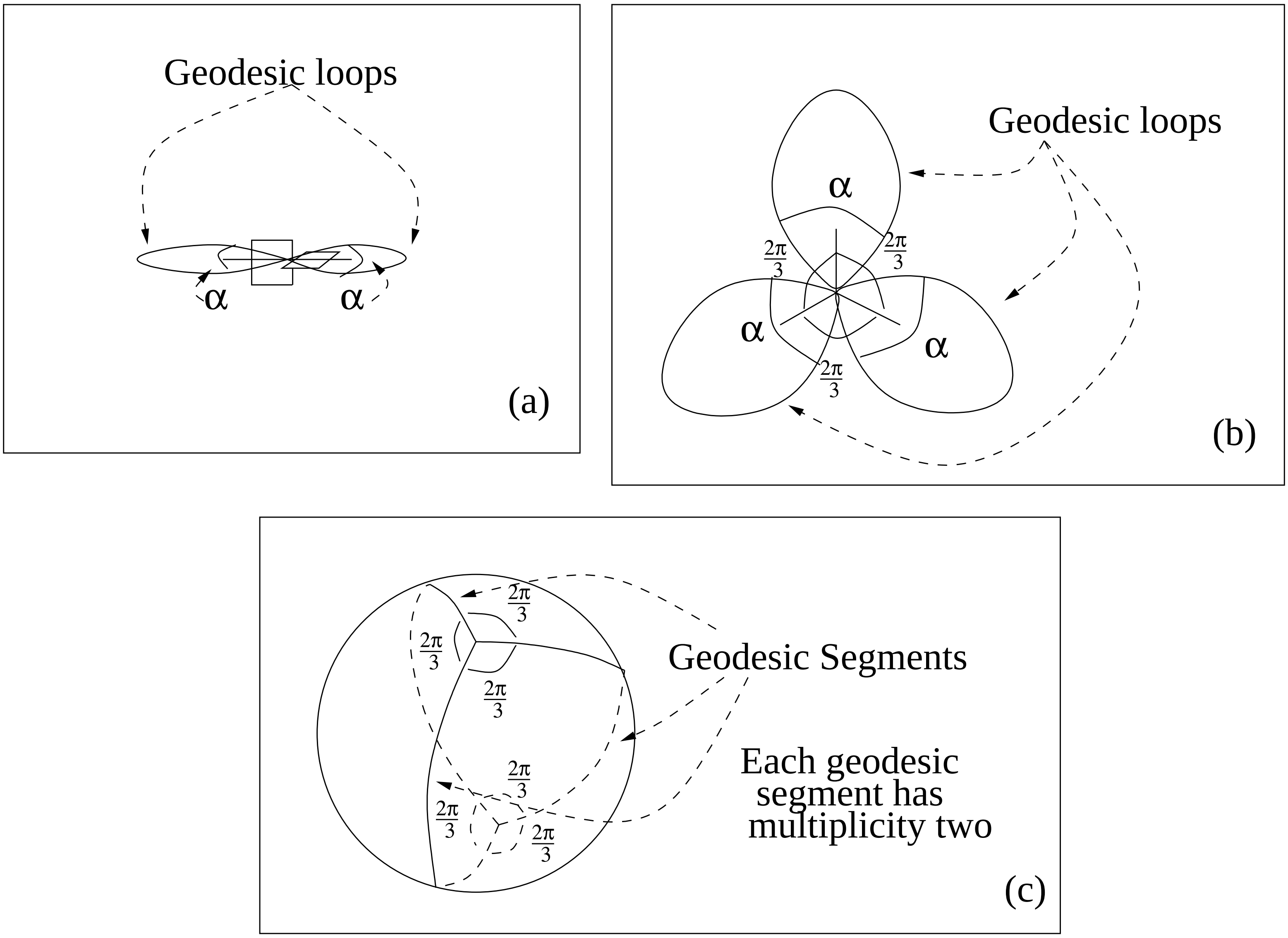


Figure 2: Examples of strongly stationary 1−cycles.

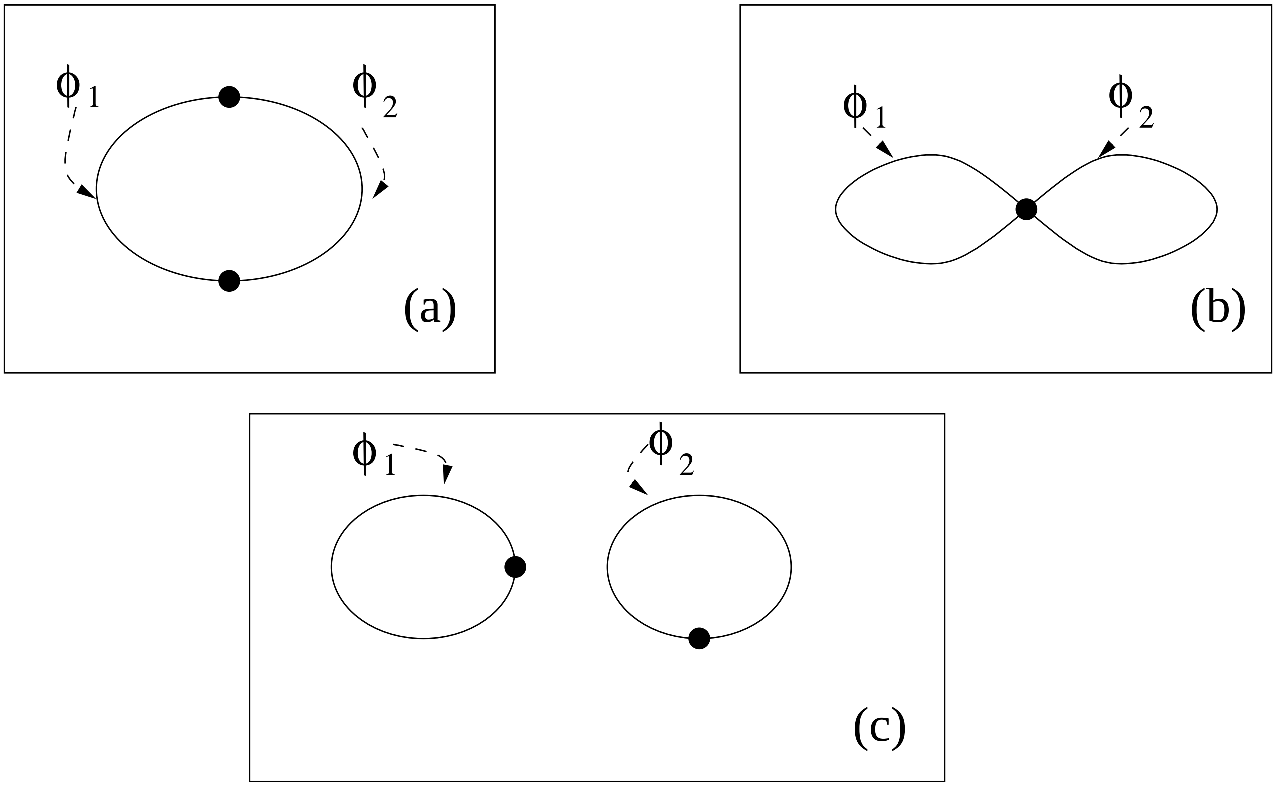


Figure 3: Examples of elements of Γ_2

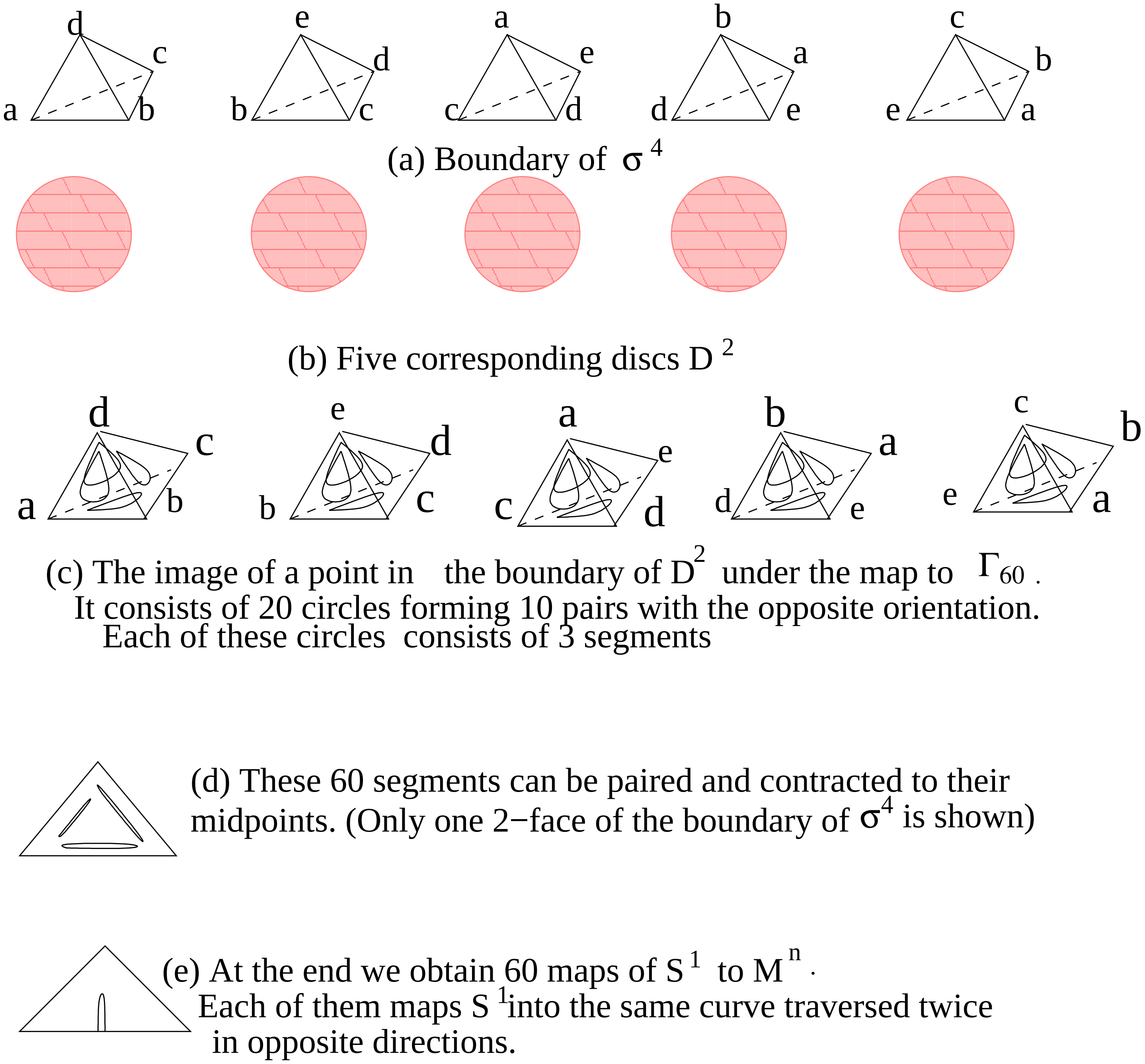
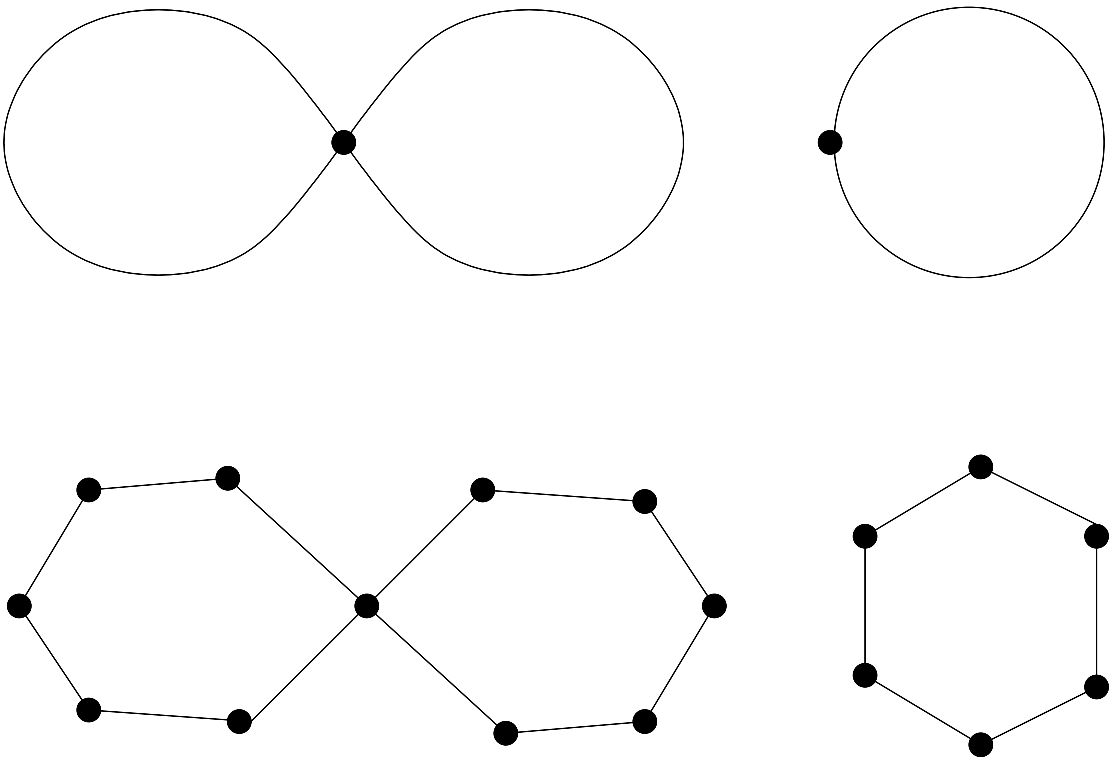

(a) Boundary of σ 4
(b) Five corresponding discs D 2
(c) The image of a point in the boundary of D 2 under the map to Γ60 .
It consists of 20 circles forming 10 pairs with the opposite orientation.
Each of these circles consists of 3 segments
(d) These 60 segments can be paired and contracted to their midpoints. (Only one 2−face of the boundary of σ 4 is shown)
(e) At the end we obtain 60 maps of S 1 to M n .
Each of them maps S 1 into the same curve traversed twice
in opposite directions.

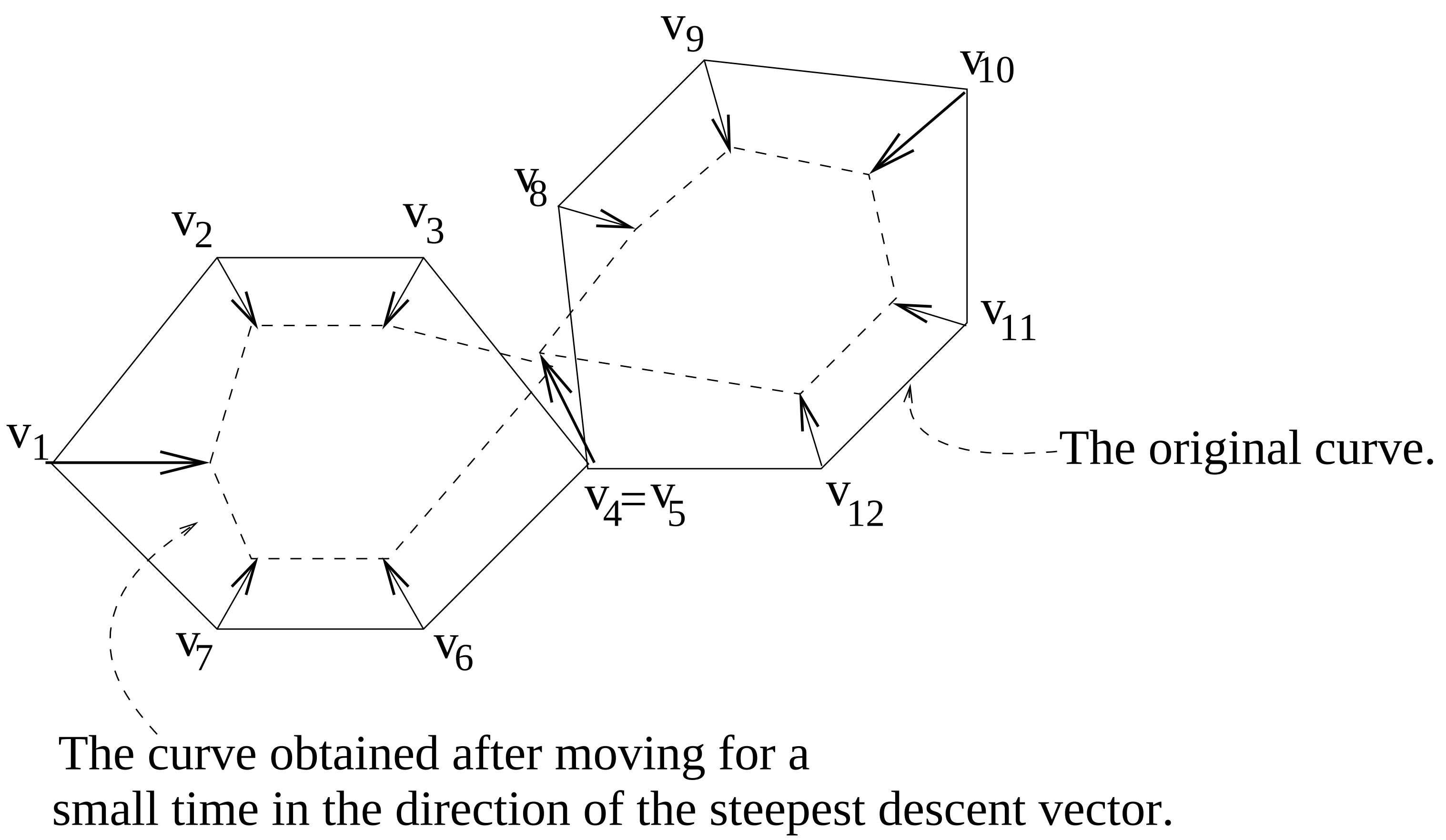
Figure 4.



Element of Γ_3^x

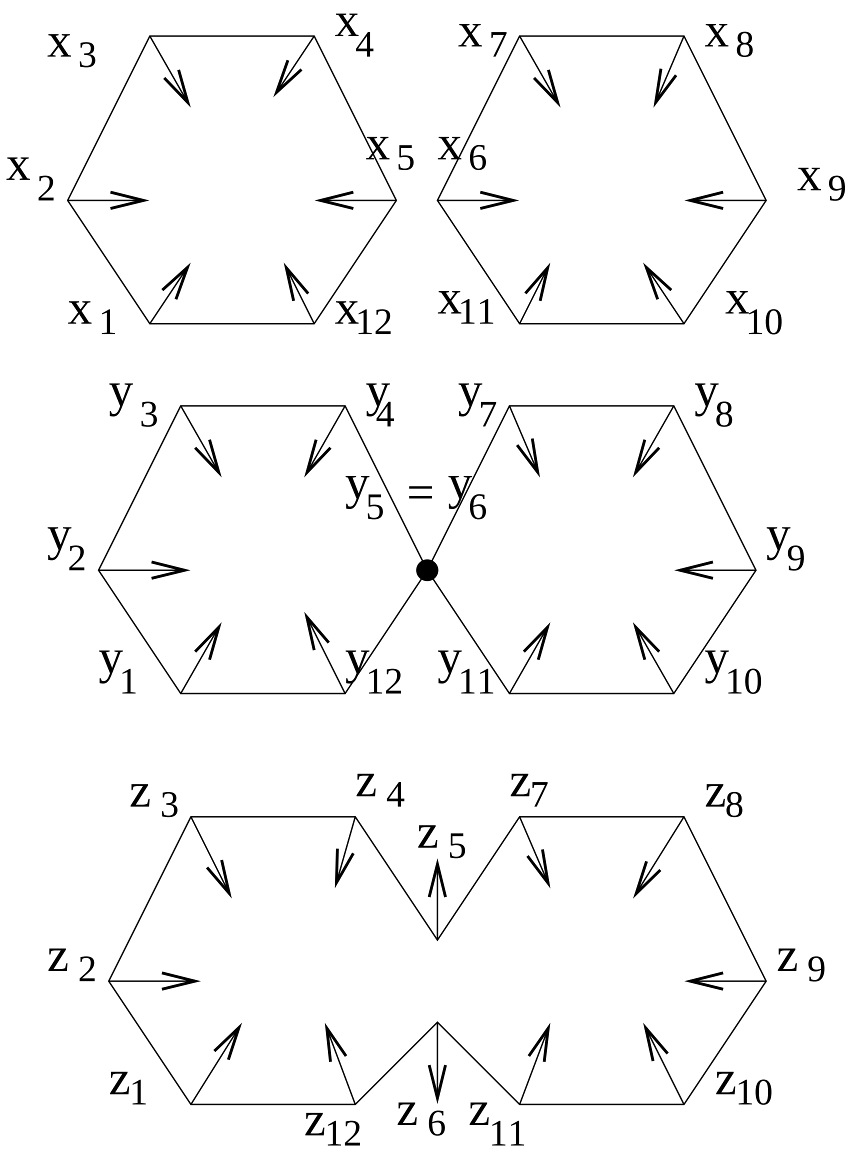
Element of $g_{3,6}^x$

Figure 5



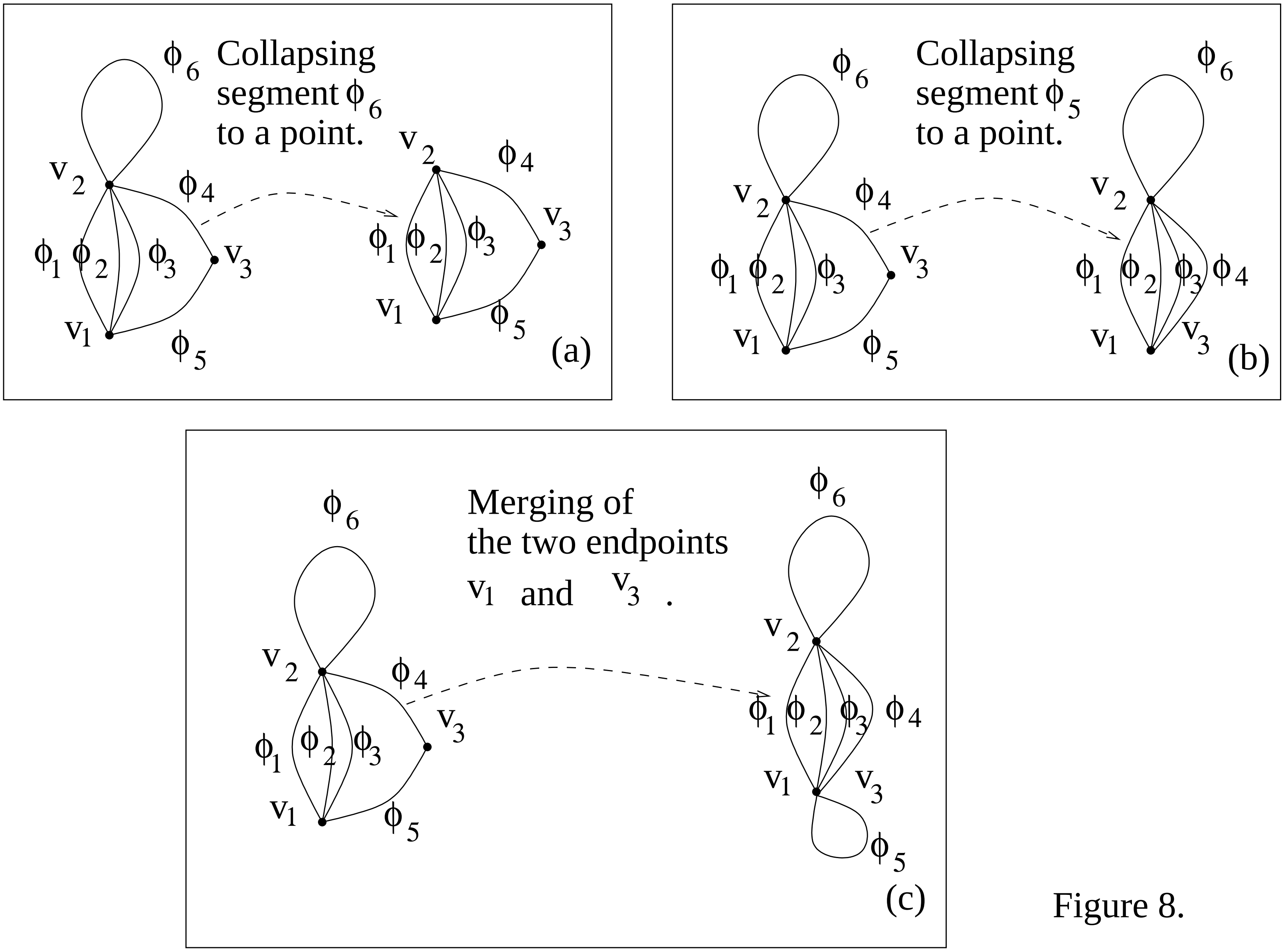
The steepest descent vector $V=(v_1, \ldots, v_{12})$

Figure 6.



Consider the elements that are close to each other in $g^{x}_{2,6}$.

This figure demonstrates that the vector of steepest descent is not continuous. Figure 7.



ϕ_6
Collapsing segment ϕ_6 to a point.
v_2
ϕ_4
ϕ_1 ϕ_2 ϕ_3
v_3
v_1
ϕ_5
(a)
Collapsing segment ϕ_5 to a point.
(b)
Merging of the two endpoints v_1 and v_3.
(c)

Figure 8.

Appendix A. A detailed proof of Lemma 3

For convenience of the reader the proof will be split in several steps.

A.1. Recall that a *deformation* of a space X into its subset A is, by definition, a continuous map g of X into itself homotopic to the identity map $X \longrightarrow X$ such that $g(X) \subset A$. (But we do not require that the restriction of g on A is the identity map.) More generally, if B is a subset of X such that $A \subset B$ we say that a map $g : B \longrightarrow A$ is a deformation in X if there exists a homotopy $G : B \times [0,1] \longrightarrow X$ such that for any $b \in B$ $G(b,0) = b$ and $G(b,1) = g(b)$. But sometimes by a deformation we will mean the whole homotopy G and not just $g(b) = G(b,1)$.

A.2. Birkhoff curve-shortening process for Γ_k^x. First, we are going to proceed as in the first step of the Birkhoff curve shortening process described in [C] or [ClCo]: Let $inj(M^n)$ denote the injectivity radius of M^n. Choose $N = [4x/inj(M^n)]+1$. Let γ be an element of Γ_k^x. Divide each of k segments γ_i of γ into N pieces of equal length by points $\gamma_i(t_{ij}), j \in \{0,1,\dots,N\}, t_{i0} = 0, t_{iN} = 1$. Consider the unique minimizing geodesic segments between $\gamma_i(t_{ij})$ and $\gamma(t_{i,j+1})$ for all j. The length of each of these segments does not exceed $inj(M^n)/4$. For any i N such geodesic segments form a piecewise geodesic $\bar\gamma_i$ connecting $\bar\gamma_i(0) = \gamma_i(0)$ with $\bar\gamma_i(1) = \gamma_i(1)$. The length of $\bar\gamma_i$ does not exceed the length of γ_i. There exists the following homotopy between γ_i and $\bar\gamma_i$: At the moment of time $\tau \in [0,1]$ we follow each of the N segments of γ_i (between $\gamma_i(t_{ij})$ and $\gamma_i(t_{i,j+1})$) from $\gamma_i(t_{ij})$ to $\gamma_i(\tau t_{ij} + (1-\tau)t_{i,j+1})$ and then make a shortcut from $\gamma_i(\tau t_{ij} + (1-\tau)t_{i,j+1})$ to $\gamma_i(t_{i,j+1})$ along the shortest geodesic. Then we reparametrize the resulting curve proportionally to its arc length. It is easy to see that the length of the curve does not increase during this homotopy. Denote the resulting homotopy by h_i. Combining all of these k homotopies h_i we obtain a homotopy between the parametrized 1-cycle γ and the parametrized 1-cycle $\bar\gamma = (\bar\gamma_1, \dots, \bar\gamma_k)$ made of k piecewise geodesics with N breaks. This homotopy depends on γ in a continuous way. Therefore we obtain a deformation of Γ_k^x into its subset $g_{k,N}^x$ defined as the set of all parametrized 1-cycles $\bar\gamma = (\bar\gamma_1, \dots, \bar\gamma_k)$ from Γ_k^x such that for any i $\bar\gamma_i$ is a piecewise geodesic made of N geodesic segments of non-zero length $\leq inj(M^n)/4$ parametrized proportionally to its arclength. Let us denote this deformation by B_N. Below we will refer to it as the *the Birkhoff deformation.* We regard $g_{k,N}^x$ as the subset of a larger set $G^x_{k,N}$ defined as the set of all elements $\gamma=(\gamma_1,\dots,\gamma_k)$ of Γ^x_k such that for any i γ_i is a piecewise geodesic made of at most N geodesic segments of non-zero length $\leq inj(M^n)/2$. In other words, the only difference between $g^x_{k,N}$ and $G^x_{k,N}$ is that we allow elements of $G^x_{k,N}$ to have somewhat longer geodesic segments.

Now we are going to prove that $g^x_{k,N}$ can be deformed into its subset $g^0_{k,N}=\Gamma^0_k$ inside Γ^x_k.

A.3 A classification of elements of Γ^x_k and $G^N_{k,x}$. Let us define an equivalence relation on Γ^x_k. For any element $\gamma=(\gamma_1,\dots,\gamma_k)$ of Γ^x_k or $G^x_{k,N}$ consider its $2k$ endpoints $\gamma_i(0),\gamma_i(1)$. We will call these points *multiple points* of γ. The set of these $2k$ points can be partitioned into J non-empty sets A_j, ($J\in\{1,\dots,k\}$), such that 1) Each set A_j contains the equal number of points of the form $\gamma_i(0)$ (for some i) and $\gamma_l(1)$ (for some l); 2) $\gamma_i(t_1)=\gamma_l(t_2)$ for some $i,l\in\{1,\dots,k\}$ and $t_1,t_2\in\{0,1\}$ if and only if $\gamma_i(t_i)$ and $\gamma_l(t_2)$ are in the same set A_j for some $j\in\{1,\dots,J\}$. The number J will be called *the number of multiple points* of γ. We will say that two 1-cycles γ and β from Γ^x_k are of the same *type* if these partitions for γ and β coincide, and the set of all i such that γ_i is constant coincides with the set of values of i such that β_i is constant.

For example, let $k=2$. In these case there will be three types of parametrized 1-cycles from Γ^x_k when neither γ_1 nor γ_2 is constant: (a) $\gamma_1(0)=\gamma_1(1)\neq\gamma_2(0)=\gamma_2(1)$ (1-cycle that consists of 2 closed curves that do not intersect at their endpoints); (b) $\gamma_1(0)=\gamma_2(1)\neq\gamma_2(0)=\gamma_1(1)$ (1-cycles that consist of one closed curve obtained by glueing together γ_1 and γ_2, where endpoints of γ_1 (and of γ_2) are different; and (c) $\gamma_1(0)=\gamma_1(1)=\gamma_2(0)=\gamma_2(1)$ (γ_1 and γ_2 are loops with the common endpoint. Such 1-cycle can be considered as either made of two closed curves or of one closed curve with the self-intersection.) There will be two types when both γ_1 and γ_2 are constant, namely, $\gamma_1\neq\gamma_2$ and $\gamma_1=\gamma_2$. For each $i=1,2$ there will also be two types corresponding to the case when γ_i is constant and γ_{3-i} is a non-constant loop: One type corresponds to the case when $\gamma_i(t)=\gamma_{3-i}(0)=\gamma_{3-i}(1)$ and the other type corresponds to the case $\gamma_i(t)\neq\gamma_{3-i}(0)=\gamma_{3-i}(1)$.

However, we will also need a stronger equivalence relation on $G^x_{k,N}\subset\Gamma^x_k$. For each $\gamma\in G^x_{k,N}$ we will consider kN geodesic segments γ_{ij} in k parametrized curves γ_i forming γ. Consider $(N-1)k$ endpoints of these geodesic segments that are not endpoints of k curves γ_i. We will call them *double points* in order to emphasize that there are exactly two geodesic segments meeting at any of these points. (However note, that it is possible that precisely two geodesic segments meet at some of the multiple points as well). We will say that two elements α, β of $G^x_{k,N}$ are of the same *type* as elements of $G^x_{k,N}$ if they are of the same type as elements of Γ^x_k and for each $i = 1, \dots, k$, $j = 1, \dots, N$ the geodesic segment α_{ij} is constant if and only if β_{ij} is constant. Also note that the type of γ as an element of $G^x_{k,N}$ and the (vectors of) positions of J multiple points and $(N-1)k$ double points on the manifold determine an element of $G^N_{k,x}$ uniquely. Therefore for any specific type of γ we can identify an element $\gamma = (\gamma_i)_{i=1}^k$ of $G^x_{k,N}$ with the corresponding element of $(M^n)^{J+(N-1)k}$, where J is the number of multiple points of γ. Note that copies of $(M^n)^{J+(N-1)k}$ corresponding to individual types can be simultaneously embedded into the ambient manifold $(M^n)^{(k+1)N}$ by the corresponding diagonal embeddings that identify endpoints of k intervals in accordance with the type.

A.3.A A partial order on types of elements of $G^x_{k,N}$. We will say that a type A is *higher* than a type B, if: 1) Multiple points of A can be obtained by merging some of the multiple points of B. (In other words the partition considered in A.3 for A is coarser than the partition for B.); 2) If for some 1-cycle β of type B, for $i = 1, 2, \dots, k$, and for some $j = 1, \dots N$ the geodesic segment β_{ij} is constant, then for any element γ of type A γ_{ij} is also constant.

The resulting relation is a partial order on the set of types. For example, the maximal types among types that correspond to cycles of non-zero length are the types with just one multiple point m, $Nk-2$ constant geodesic segments, and just two non-constant geodesic segments. These two segments correspond to the minimal geodesic connecting m with the only double point, d, and traverse this geodesic in opposite directions.

A.4. Some general remarks about deformations of $g^x_{k,N}$ that will be constructed below. These deformations will consist of finitely many steps. Each of these steps constitute either the Birkhoff deformation described in A.2, or will be a deformation inside $G^x_{k,N}$. In the last case in order to describe the deformation we need to describe the trajectories of individual multiple and double points. More formally, during a deformation of the last type we will be deforming each individual cycle γ as an element of $(M^n)^{(N+1)k}$. Each multiple or double point will move along trajectories obtained as a projection of trajectories of a vector field defined in an open set in $(M^n)^{(N+1)k}$ that contains the union of all diagonnaly embedded copies of $(M^n)^{J+(N-1)k}$ corresponding to various types of elements of $G^x_{k,N}$.

We will construct this vector field in such a way that will not allow the type of the elements of $G^x_{k,N}$ to change during these stages of the deformation, but at the very end of the deformation, when the trajectory of the flow hits $G^0_{k,N}$, and the length of the element becomes zero. This is achieved by defining the components of the vector fields that correspond to the individual multiple and double points of γ so that in the situation when the distance between two distinct multiple and/or double points becomes small, they will move along the trajectories of the same smooth vector field on M^n. (Here we are talking only about pairs of points the collision of which can change the type.) Moreover, since the type of any 1-cycle from $g^x_{k,N}$ regarded as an element from Γ^x_k cannot change during the Birkhoff deformation (since all multiple points remain unchanged), Γ^x_k types of 1-cycles remain unchanged through the whole deformation until its very last moment. We will not need this feature of our construction in the proof of Lemma 3, but it will turn out to be convenient for the next section.

However, we can encounter the following problem: We need to ensure that the distance between any two (double or multiple) points that should be connected by a geodesic segment is less than $inj(M^n)$. We will resolve this technical complication in the following way. Our choice of N ensures that at the beginning these distances do not exceed $inj(M^n)/4$ for any $\gamma \in g^x_{k,N}$. Therefore we can deform our parametrized 1-cycles using a flow that will be constructed below for a certain safe amount of time which is sufficiently small to ensure that lengths of segments grow by not more than $inj(M^n)/4$. Then we stop and apply the Birkhoff deformation with N breaks B_N again. That is, we forget that the k curves forming an element of $G^x_{k,N}$ are already piecewise geodesics with N breaks, divide them into N equal pieces (of length $\leq x/N \leq inj(M^n)/4$) by $N-1$ points and replace the curve by another piecewise geodesic consisting of N geodesic segments of length $\leq inj(M^n)/4$ with vertices in these points. (Of course, this replacement is made using the length non-increasing homotopy described in A.2 that was used in the definition of the Birkhoff deformation.) At this point, in principle, the type of γ as an element of $G^x_{k,N}$ can change because new double points can merge in a different pattern from what was before the Birkhoff stage. (Note however that the Γ^x_k type remain unchanged since B_N does not affect multiple points.) Now we are ready to continue the deformation using the same flow again, etc... The vector fields and the times of these deformations will be chosen so that at each stage of the deformation every element of $G^x_{k,N}$ will become shorter by at least a certain $\delta = \delta(M^n, N, k, x) > 0$. Therefore we will need only a finite number of stages to reach $G^0_{k,N}$.

A.5. The direcion of the steepest descent. We will start the construction of the flow from the following observation: Since there is no strongly stationary 1-cycle of positive length $\leq x$, then for any piecewise geodesic 1-cycle γ from $G^x_{k,N}$ there exists a system of vectors at all multiple and double points such that a small deformation of γ (regarded as an element of $(M^n)^{J+(N-1)k}$) in the direction of these vectors leads to an element of $G^x_{k,N}$ of the same type but of a smaller length. These vectors are constructed as follows: Any multiple point corresponds to a set A_j of the partition. The vector at this point is calculated as $\Sigma_{\gamma_l(t_i)\in A_j; t_i=0 \ or \ 1} v_l(t_i)$, where $v_l(t_i)$ is the unit vector tangent to γ_l at t_i directed from the multiple point. If we will regard all Nk geodesic segments as curves in M^n, then each double point is adjacent to two geodesic segments of the curve (with the exception of the case, when this double point is connected by a sequence of geodesic segments of zero length with a multiple point. In this last case the double point coincides with the multiple point, and we *define* the component of v corresponding to this double point to be equal to the already determined component of v corresponding to the multiple point.) The component of v at this double point is calculated as the sum of two unit vectors tangent to two geodesic segments that meet at this point and are directed from it. Our assertion now follows directly from the first variation formula for the length functional. We will call the system of $J+(N-1)k$ tangent vectors of M^n *a deformation vector* for γ and will denote it by $v(\gamma)$. Note that $v(\gamma)$ is the collection of zero vectors if and only if γ is a strongly stationary 1-cycle.

The first variation of the length of γ in the direction of $v(\gamma)$ is equal to $-\|v(\gamma)\|^2$, where $\|v(\gamma)\|^2$ is calculated as follows: When we deal with J components of v corresponding to multiple points we just sum their squares. We are going to say that a pair of double points *merges* if they are connected by a sequence of *constant* geodesic segments. We define a *cluster* of double points as a maximal set of double points such that each pair of them merges. We say that a cluster is *negligible* if one of double points in the cluster is connected by a sequence of *constant* segments with a multiple point (so geometrically, all double points in this cluster coincide with the multiple point). Our definition of v implies that components of v corresponding to double points in a cluster point are equal. When we calculate $\|v(\gamma)\|^2$ we *by definition* disregard all double points in each negligible cluster and count the squared norm of the component of v corresponding to all double points in a cluster only once for each non-negligible cluster. In other words we define $\|v(\gamma)\|^2$ as $\Sigma_{m_i;i=1,\dots J}\|v(\gamma)(m_i)\|^2 + \Sigma_{Non-negligible\ clusters\ of\ double\ points}\|v(\gamma)(d_i)\|^2$, where the first summation is over the set of all multiple points and the second summation is over the set of all non-negligible clusters of double points.

A.6. The deformation vector for γ can be used to decrease the length for all γ_* sufficiently close to γ. Unfortunately, the dependence of $v(\gamma)$ on γ is not continuous. This happens because the type of element of $G^x_{k,N}$ changes in a discontinuous manner. Yet, it is easy to see that for any $\gamma \in G^x_{k,N}$ there exists a sufficiently small open neighborhood U of γ in $G^x_{k,N}$ and a positive μ such that for any $\gamma_* \in U$ a sufficiently small deformation of γ_* in the direction of the deformation vector $T^{\gamma_*}_{\gamma}(v(\gamma))$ defined below decreases the length of γ_*, and the first variation of the length in the direction of $T^{\gamma_*}_{\gamma}(v(\gamma))$ is less than or equal to $-\|v(\gamma)\|^2/2$. The above deformation vector, $T^{\gamma_*}_{\gamma}(v(\gamma))$ is obtained from $v(\gamma)$ by the parallel transport of all $J+\Sigma_i(N_i-1)$ vector components of $v(\gamma)$ along the shortest geodesics connecting the vertices of γ with the corresponding vertices of γ_*.

More precisely, one first chooses U so small that: 1) different multiple or double points of γ cannot merge in U; 2) For any $\gamma_* \in U$ any of its multiple points has the unique closest multiple point of γ, and any double point of γ_{*i} has the unique closest double or multiple point of γ_i at the distance not exceeding $inj(M^n)/4$. But note that, in principle, each multiple point of γ can split into two multiple points (that can be connected or not connected by a geodesic segment) or into a pair multiple point - double point for an arbitrarily small U. Also note that if $N_i < N$ then each double point of γ_i can bifurcate into two distinct double points connected by a very short geodesic. If one of k segments γ_i is a constant geodesic loop, then the corresponding multiple point can bifurcate into a pair of points that consists of the multiple point and a double point that is very close to the multiple point. These points are connected by two oppositely oriented copies of the shortest geodesic, together forming a short piece-wise geodesic loop based at the multiple point. Finally note that a finite number of bifurcations of these types can occur simultaneously. So, the dimension of $T_{\gamma}^{\gamma_*}(v(\gamma))$ can be greater than the dimension of $v(\gamma)$. But condition 2) in the definition of U implies that even if a multiple or a double point of γ bifurcates into a finite number of (multiple and/or double) points we know unambigiously how to define the corresponding component of $T_{\gamma}^{\gamma_*}(v(\gamma))$ for each of them: we just perform the parallel transport of the corresponding component of $v(\gamma)$ along the (unique) shortest geodesic.

Now the assertion immediately follows from the continuity on U of the first variation of the length in the direction of the field $X(\gamma_*) = T_{\gamma}^{\gamma^*}(v(\gamma))$. This continuity follows from the first variation formula for the length functional. One just needs to perform easy calculations verifying this continuity for all cases of elementary mergers. That is, it is necessary to consider the particular cases, when $\gamma_i \longrightarrow \gamma$, where all γ_i are of the same type, which can be obtained from (the type of) γ by either 1) a splitting of a multiple point into two multiple points (connected or not connected by a very short geodesic segment converging to the point) or into the pair "multiple point-double point" (on one of the segments adjacent to the multiple point); or 2) a splitting of a double point into two double points connected by a very short geodesic (converging to the double point). The general case of this formula follows by induction. We will omit the details of this straightforward verification.

A.7. After these preliminaries we are going to prove that:

A. There exists a sufficiently small positive $\tau_* \leq x$ such that $g_{k,N}^{\tau_*}$ can be deformed to Γ_k^0; and

B. For any positive $\tau \leq x$ there exists a deformation of $g_{k,N}^x$ to $g_{k,N}^{\tau}$ inside $G_{k,N}^x$.

In fact, as it was already noted, we are going to construct the deformation by first defininining a vector flow on an open subset of $(M^n)^{(N+1)k}$, that includes the image of $G_{N,k}^x$ under the embedding discussed above. This flow will be the same for both A and B. So the division of our deformation into these two parts is somewhat artificial. Yet we encounter different difficulties in these two situations: When the length is small, our main problem will be the lack of compactness at zero length of the space $G^x_{N,k} \setminus G^0_{N,k}$, and we need to prove that our flow decreases the length with a speed bounded from zero by a constant. When the length is large, we do not want the distances between double points that must be connected by a geodesic segment, to become large, so from time to time we stop and perform the Birkhoff deformation.

A.8. In order to prove A we would first like to establish a positive lower bound for $\|v(\gamma)\|$ for all $\gamma \in G^{\tau_0}_{k,N} \setminus G^0_{k,N}$ for a sufficiently small positive τ_0 . The key idea is to observe that this statement will be true for the Euclidean space R^n instead of M^n : Assume that there exists a sequence of 1-cycles γ_i in R^n made of at most Nk straight line segments such that $\|v(\gamma_i)\| \longrightarrow 0$. Rescale γ_i in R^n so that the maximal length of an edge equals to one. (This does not affect $\|v(\gamma_i)\|$.) Choose a convergent subsequence. Then its limit must be a non-trivial stationary 1-cycle in R^n of length ≥ 1 . (Here we must check what happens with $\|v\|$ when an edge collapses to a point in the limit. It is easy to see that $\|v(\lim_{i\longrightarrow\infty}(\gamma_i)\| \leq 2\limsup_{i\longrightarrow\infty}\|v(\gamma_i)\|$ in the situation, when we have a sequence of γ_i of the same type, and exactly one segment of γ_i collapses to a point in the limit. The number of such collapses is bounded from above by $Nk-1$. Therefore the norm of the deformation vector of the limit 1-cycle will be zero.) But it is very easy to see that there are no stationary 1-cycles in R^n . So, we obtain a contradiction thereby proving the existence of a uniform positive lower bound for $\|v(\gamma)\|$ for all parametrized 1-cycles that consists of at most Nk straight line segments in R^n .

If $\tau_0 = \tau_0(M^n, N, k)$ is sufficiently small, then any parametrized 1-cycle from $G^{\tau_0}_{k,N}$ splits into several connected components contained in very small balls in M^n . Applying the inverse of the exponential map we obtain "almost" a 1-cycle in the tangent space to M^n with "almost" the same angles. Now the existence of a uniform positive lower bound for the norm of the deformation vectors of elements of $G_{k,N}$ for R^n implies the existence of such uniform lower bound for all 1-cycles from $G^{\tau_0}_{k,N}$.

A.9. Construction of a deformation of $g^{\tau_0}_{k,N}$ into Γ^0_k. Now it is easy to find a countable set $\{\gamma_l\} \subset G^{\tau_0}_{k,N} \setminus \Gamma^0_k$, a locally finite covering of $G^{\tau_0}_{k,N} \setminus \Gamma^0_k$ by open balls U_l centered at γ_l and a subordinate partition of unity that can be used to obtain a continuous function ϕ assigning to every element γ of $G^{\tau_0}_{k,N} \setminus \Gamma^0_k$ a system of tangent vectors to M^n at each of its multiple or double points such that the variation of length of the cycles in direction of $\phi(\gamma)$ is bounded from below by a positive constant δ. (In order to obtain a formal proof of the last assertion we just need to establish the continuity of the first variation of the length. This will be done in A.10.)

We construct this γ_l and U_l inductively with respect to the partial order on types of elements of $G^x_{k,N}$ introduced in section A.3.A above. We start from elements of $G^x_{k,N} \setminus G^0_{k,N}$ of the highest possible type that form strata of a high codimension that are closed in $G^x_{k,N} \setminus G^0_{k,N}$. Construct a locally finite covering (in $G^x_{k,N}$) of the union of these strata, so that all centers of open balls forming the covering are on the considered strata. Automatically an open neighborhood of the union of these strata will be covered. Then we proceed to strata corresponding to types of the second highest order. Points of closure of the union of these strata in $G^x_{k,N} \setminus G^0_{k,N}$ that are not in these strata are in strata corresponding to the highest type, and were already covered. Therefore we can choose the covering so that no open ball from this covering intersects an open neighborhood of strata corresponding to the highest types that were previously covered. Also the centers of all balls should be on the strata that are being covered. We continue inductively in this way until we cover the whole $G^x_{k,N} \setminus G^0_{k,N}$. On each step of the inductive procedure we consider the union strata corresponding to the maximal types that were not yet covered. A neighborhood of the union of all strata corresponding to higher types was already covered on previous steps of the inductive procedure. So we complete the covering of of the union of strata that are being considered on the current step by adding open balls centered at the considered strata that have empty intersections with a neighborhood of the union of all already covered strata corresponding to all higher types.

After the covering is completed, and a subordinate partition of unity is chosen, we define $\phi(\gamma)$ as the weighted sum of $T^{\gamma}_{\gamma_i}(v(\gamma_i))$ over the set of indices i such that $\gamma \in U_i$. (Recall that components of $T^{\gamma}_{\gamma_i}(v(\gamma_i))$ are obtained by the parallel transport of the corresponding components of $v(\gamma_i)$ along the shortest geodesics between corresponding multiple or double points of γ_i and γ. Of course, U_i should be sufficiently small in order for this definition to be unambigious, as we explained above.) The weights are equal to the corresponding functions from the partition of unity. This construction provides us with the flow Φ_t that deforms $g^{\tau}_{k,N}$ to Γ^0_k in a finite time for each sufficiently small τ (as we will see below).

A.10. The type cannot change during this deformation; the first variation of the length is continuous. Let p_1 , $p_2 \in \gamma \in G^x_{k,N} \setminus G^0_{k,N}$ be either two multiple points, or a multiple point and a double point connected by a segment, or two double points connected by a segment. Note that if the distance between p_1 , p_2 of γ is very small, then γ is in a small neighborhood of a stratum corresponding to a higher type of cycles, where p_1 and p_2 merge into one point p . If this neighborhood is sufficiently small, then *all* balls of the covering that cover γ are centered at strata corresponding to higher types, where p_1 and p_2 are merged into one point p . But then p_1 and p_2 will be deformed (for some period of time) using the same vector fields on M^n . This will be happening all the time while they will be sufficiently close to each other. But since different integral trajectories of a smooth vector field do not intersect, p_1 and p_2 cannot merge at least until the moment of the deformation, when the total length of the 1-cycle becomes zero.

Also, observe that in the considered situation (when γ is close to a stratum corresponding to a higher type) the first variation of the length in the direction of the vector field that we constructed will be equal to a linear combination of variations in the direction of vector fields of the form $T^{\gamma}_{\gamma_l}(v(\gamma_l))$, considered in A.6. (Here γ_l are located in strata corresponding to higher types, where p_1 and p_2 merge into one point. The coefficients in the linear combination will be the corresponding functions from the partition of unity.) Therefore we can prove the continuity of the first variation of length with respect to γ as in A.6.

A.11. Note that for any $\gamma \in G^{\tau}_{k,N}$ $\Phi_t(\gamma)$ is defined only until the moment of time $t(\gamma)$, when the length of $\Phi_t(\gamma)$ will become zero. But one can extend the domain of definition of Φ_t by defining $\Phi_t(\gamma) = \Phi_{t(\gamma)}(\gamma)$ for $t > t(\gamma)$. More precisely, we take $\tau_* = \min\{x, \tau_0, \delta, inj(M^n)/4\}$, where τ_0 is as in 3.8, and δ is the lower bound of the speed of decrease of the length introduced at the beginning of 3.9, and just follow the flow until we hit Γ^0_k . It is clear that: 1) For any element $\gamma \in g^{\tau}_{k,N}$ we will reach Γ^0_k in time $t(\gamma) \leq 1$; 2) Since the total length of γ decreases, the distance between any two points on M^n that should be connected by the shortest geodesic in order to obtain $\Phi_t(\gamma)$ does not exceed $inj(M^n)/4$. Therefore

$\Phi_t(\gamma)$ is unambigiously defined. (Recall that we move multiple and double points of γ along trajectories of vector fields determined by the corresponding components of $\phi(\gamma)$. In order to obtain $\Phi_t(\gamma)$ we connect these points by the shortest geodesics.) 3) $t(\gamma)$ depends on γ continuously (by virtue of the implicit function theorem. In order to apply the implicit function theorem we need to establish the continuous differentiability of the length as a function of γ, but it is equivalent to the continuity of the first variation of the length of γ in the direction of the vector field that was established in the previous subsection.)

Therefore the map assigning to γ the point $\Phi_{t(\gamma)}(\gamma) \in \Gamma_k^0$, where the trajectory of the flow reaches Γ_k^0 is continuous, and is the deformation of $G_{k,N}^{\tau}$ to Γ_k^0.

A.12. It remains to prove the existence of a deformation of $g_{k,N}^x$ to $g_{k,N}^{\tau_*}$ inside Γ_k^x. Use the compactness of the closure S of $G_{k,N}^x \setminus G_{k,N}^{\tau_*/2}$ to find a finite open covering of S by open neighborhoods U_l of $\gamma_l \in S$ such that for any $\gamma_* \in U_l$ the first variation of the length of γ in the direction of $T_{\gamma_l}^{\gamma_*}(v(\gamma_l))$ does not exceed $-\|v(\gamma_l)\|^2/2$. (Recall that we have proved the existence of an open neighborhood U with this property for any $\gamma \in G_{k,N}^x \setminus \Gamma_k^0$.) Using a subordinate partition of unity $\alpha_l(\gamma)$ define a vector field $\phi(\gamma)$ by the formula $\phi(\gamma) = \Sigma_l \alpha_l(\gamma) T_{\gamma_l}^{\gamma}(v(\gamma_l))$, where we perform the summation only over indices l such that $\gamma \in U_l$.

Here the construction of this open covering is similar to that in section A.9: We construct it inductively starting from the strata that correspond to the highest type and then proceed to cover strata corresponding to lower types. On each step we add only open balls centered at points on the considered strata that do *not* intersect with an open neighborhood of the already covered strata corresponding to higher types. As the result the $G_{k,N}^x$ type will not be changing during the considered stage of the deformation, and the proof of this fact coincides with the proof in section A.10 above almost verbatim.

Rescale $\phi(\gamma)$ by a continuous function equal to zero on $G_{k,N}^{\tau_*/2}$ and to one on $G_{k,N}^{\tau_*}$. Denote the resulting vector field by $\psi(\gamma)$. Let $t_* = inj(M^n)/(16k)$. Consider the flow $\Psi_t(\gamma)$ defined for all $\gamma \in g_{k,N}^x$ at $t \in [0, t_*]$ and determined by the vector field $\psi(\gamma)$. Our choice of t_* guarantees that $\Psi_t(\gamma)$ will be in $G_{k,N}^x$. (In other words, the distance between any pair of points that need to be connected by a geodesic segment will not exceed $inj(M^n)/2$.) Observe that for any $\gamma \in g_{k,N}^x \setminus G_{k,N}^{\tau_*}$ the difference between the length of γ and the length of

$\Psi_{t_*}(\gamma)$ will be at least δt_*, where $\delta = \frac{1}{2}\min_{i=1}^{l} \|v(\gamma_l)\|^2$.

Now recall that the Birkhoff curve shortening process provides us with the deformation B_N of Γ_k^x into $g_{k,N}^x$. The restriction of B_N to $G_{k,N}^x \subset \Gamma_k^x$ is a deformation of $G_{k,N}^x$ into $g_{k,N}^x$ inside Γ_k^x. Apply B_N. The composition of B_N and Ψ_{t_*} is a curve-shortening deformation of $g_{k,N}^x$ into $g_{k,N}^{\max\{x-\delta t_*,\tau_*\}}$ inside Γ_k^x.

Now we can apply Ψ_{t_*} and then B_N again and again, etc. Let $K = [\frac{1}{\delta t_*}] + 1$. It is easy to see that $(B_N\Psi_{t_*})^K$ is the required deformation of $g_{k,N}^x$ to $g_{k,N}^{\tau_*}$ inside Γ_k^x. QED.

Observation. We would like to mention again that the defined in A.3 (or 3.8) type of elements of Γ_k^x does not change during the considered deformation until possibly the very last moment, when the length becomes zero.

Appendix B. Almgren correspondence without the local triviality assumption

Recall that at the beginning of the proof of Lemma 3 we introduced $N = N(M^n, x) = [4x/inj(M^n)] + 1$, the spaces $g_{k,N}^x$ and $G_{k,N}^x$ made of parametrized 1-cycles of length $\leq x$ formed by k piecewise geodesics made of at most N geodesic segments of length $\leq inj(M^n)/4$ and $\leq inj(M^n)/2$, correspondingly, parametrized proportionally to the arclength. We also defined the curve-shortening Birkhoff deformation of Γ_k^x into $g_{k,N}^x$. Further recall that $g_{k,N}^x$ and $G_{k,N}^x$ can be regarded as subsets of M^{kN}. It is easy to prove that there exists a subset $\bar{g}_{k,N}^x$ of M^{kN} containing $g_{k,N}^x$ and contained in $G_{k,N}^x$ that can be triangulated. (The shortest formal way to prove the last assertion is the following: Approximate the Riemannian metric on M^n in C^3-topology by an analytic Riemannian metric so that the distances on the resulting Riemannian manifold $\bar{M}^n$ do not exceed corresponding distances on M^n. Now observe that metric balls of radius $\leq inj(M^n)/2$ on $\bar{M}^n$ are subanalytic sets, the restriction of the distance function to such metric balls is a subanalytic function, and that according to a well-known theorem of H. Hironaka subanalytic sets are triangulable (cf. [B] for more details. See also [BM] for the definition and basic properties of subanalytic sets and function, including the proof of the mentioned theorem of H. Hironaka.) Therefore we can define $\bar{g}_{k,N}^x$ as $g_{k,N}^x$ but using the distance function on $\bar{M}^n$ instead of the distance function on M^n.) It is easy to see that one can triangulate $\bar{g}_{k,N}^x$ so that the type of parametrized 1-cycles is constant on every simplex of the triangulation. Therefore we can take $x = \max_{y \in K} l(A(y))$, compose B_N with A, and take a simplicial approximation $\bar{A}$ of the resulting composition $B_N A : K \longrightarrow \bar{g}^x_{k,N}$. Now we can consider the quotient $X_{\bar{A}}$ defined as above. It is easy to see how to triangulate $X_{\bar{A}}$. Therefore we can proceed as above, and consider the corresponding singular chain in M^n. It is clear that if $K = S^{m-1}$, then this chain is a cycle, and this cycle represents $0 \in H_{m-1}(M^n)$ if and only if A is contractible. A small technical complication that arises here is the following: Assume that we apply the Almgren correspondence to a map $B : D^m \longrightarrow \Gamma^x_k$ and to the restriction A of B to $S^{m-1} = \partial D^m$. Then the values of x defined for these two mappings will , in general, be different. Therefore the cycle in M^n corresponding to A will not, in general, coincide with the boundary of the chain corresponding to B. Yet, it is easy to construct a homology between these two cycles.

In the case of a map of a polyhedron into $Z_{(k)}$ one can consider a sufficiently fine subdivision of the polyhedron K. For any simplex of this subdivision the restriction of our map onto this simplex lifts to Γ_k. Then we can proceed as in the parametrized case (for this simplex). Finally sum the resulting chains over all simplices of the triangulation.

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