We consider the semlinear wave equation

$$u_{tt} - \Delta u + f(u) = 0 \qquad \text{in } [0,T]\times\mathbb{R}^n \tag{0.1}$$

with initial data

$$u(0,x) = u_0(x), \qquad u_t(x,0) = u_1(x). \tag{0.2}$$

We assume the functions f is smooth.

We will generally assume that u is a scalar function. In fact, many of our arguments are valid with very little change in the case where u takes values in $\mathbb{R}^k$ or $\mathbb{C}^k$ for some $k \geq 2$.

We will use the notation

$$\Box u = u_{tt} - \Delta u.$$

1. CONSERVED QUANTITIES

First note that the semilinear equation (0.1) is the Euler-Lagrange equation for the functional

$$\mathcal{L}[v] := \int_0^T \int_{\mathbb{R}^n} \left(-\frac{1}{2} v_t^2 + \frac{1}{2}|Dv|^2 + F(v) \right) \, dx\, dt.$$

where $F' = f$. The Lagrangian is invariant with respect to space and time translations, and by Noether's Principle this implies that solutions of the equation obey certain conservation laws.

In particular, by multiplying (0.1) by u_t and rearranging, we find that

$$\frac{d}{dt}\left(\frac{1}{2}(u_t^2 + |Du|^2) + F(u) \right) = \operatorname{div}(u_t Du) + u_t(\Box u + f(u)) \tag{1.1}$$

Thus for a solution of the (0.1),

$$\frac{d}{dt}\left(\frac{1}{2}(u_t^2 + |Du|^2) + F(u) \right) = \operatorname{div}(u_t Du). \tag{1.2}$$

This is interpreted as expressing the conservation of energy. A second conservation law, interpreted as conservation of momentum, could be derived by multiplying the equation by u_{x_i} and arguing as above. This is less useful and we will not go into it here.

Let us write

$$e(u) = \frac{1}{2}(u_t^2 + |Du|^2) + F(u), \quad E(u) = \int_{\mathbb{R}^n} e(u)\ dx.$$

It follows from (1.2) that for a smooth, compactly supported solution of (0.1),

$$\frac{d}{dt} E(u) = \int_{\mathbb{R}^n} \frac{d}{dt} e(u)\, dx \ = \ \int_{\mathbb{R}^n} \operatorname{div}(u_t\, Du)\, dx \ = \ 0. \tag{1.3}$$

Note in particular that this holds for the linear wave equation.

The above fact can be "localized"; this is part of the content of the following lemma

Notation: we fix $(t_0, x_0) \in \mathbb{R}. \times \mathbb{R}$, and we write

$$K := \{(t,x) \ : \ |x - x_0| < t_0 - t, t > 0\}. \tag{1.4}$$

For $s < t$ we write $K_{s,t} := \{(\sigma, x) \in K : s < \sigma < t\}$. We also write

$$D_t := \{x : |x - x_0| < t_0 - t\}$$

and $R_{s,t} := \partial K_{s,t} \setminus (D_s \cup D_t)$. We will sometimes write for example $K(t_0, x_0)$, $K_{s,t}(t_0, x_0)$ etc when we need to keep track of the vertex of the cone,

Lemma 1. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a smooth, bounded function Suppose also that $u \in H^2_{loc}([0,T] \times \mathbb{R}^n)$ solves*

$$\Box u + f(u) = h \qquad \text{in } [0,T] \times \mathbb{R}^n \tag{1.5}$$

for some function $h \in L^2_{loc}([0,T] \times \mathbb{R}^n)$.

Then for every $0 \le s < t$,

$$\int_{D_t} e(u)\ dx - \int_{D_s} e(u)\ dx \le \int_{K_{s,t}} u_t h\ dx\, dt - \frac{1}{\sqrt{2}} \int_{R_{s,t}} F(u). \tag{1.6}$$

Here the equation (1.5) is assumed to hold in the sense that both sides belong to $L^2([0,T] \times \mathbb{R}^n)$, and they are equal a.e..

Proof. Note that the formal derivation we gave above of (1.1) is justified if $u \in H^2([0,T] \times \mathbb{R}^n)$. We rewrite (1.1) in the form

$$\begin{aligned} \mathrm{div}_{t,x}(e(u), -u_t Du) &:= (\partial_t, \partial_{x_1}, \dots, \partial_{x_n}) \cdot (e(u), -u_t Du) \\ &= u_t(\Box u + f(u)). \end{aligned}$$

Thus in the present case

$$\mathrm{div}_{t,x}(e(u), -u_t Du) = u_t h.$$

We fix $s < t$ and integrate this identity over $K_{s,t}$. By the divergence theorem (in the (t,x) variables),

$$\begin{aligned} \int_{K_{s,t}} u_t\, h\ dx\, dt &= \int_{K_{s,t}} \mathrm{div}_{t,x}(e(u), -u_t Du)\ dx\, dt \\ &= \int_{\partial K_{s,t}} (e(u), -u_t \nabla u) \cdot \nu_{t,x}\ dx\, dt. \end{aligned} \tag{1.7}$$

where $\nu_{t,x}$ denotes the spacetime unit normal to $\partial K_{s,t}$. Clearly $\nu_{t,x}$ has the form

$$\nu_{t,x} = (1, 0, \dots, 0) \text{ on } D_t, \qquad \nu_{t,x} = (-1, 0, \dots, 0) \text{ on } D_s$$

On $R_{s,t}$, it is not hard to check that

$$\nu_{t,x} = \frac{1}{\sqrt{2}}(1, \nu_x), \qquad \nu_x := \frac{x - x_0}{|x - x_0|}.$$

(To see this, note for example that $R_{s,t}$ is the zero level set of the function $\zeta(x,t) := |x - x_0| - (t_0 - t)$, so the normal to $R_{s,t}$ is parallel to the spacetime gradient of ζ.) When we use these facts to rewrite the boundary integral on the right-hand side of (1.7), we find that

$$\begin{aligned} \int_{K_{s,t}} u_t\, h\ dx\, dt = \int_{D_t} e(u)\ dx &- \int_{D_s} e(u)\ dx \\ &+ \frac{1}{\sqrt{2}} \int_{R_{s,t}} e(u) + \nu_x \cdot \nabla u u_t. \end{aligned} \tag{1.8}$$

Note in addition that at every point on $R_{s,t}$,

$$e(u) + \nu_x \cdot \nabla u u_t \ge \frac{1}{2}(u_t^2 + |\nabla u|^2) + F(u) - |\nabla u|\ |u_t| \ge F(u),$$

since $|\nabla u|\, |u_t| \le \frac{1}{2}(u_t^2 + |\nabla u|^2)$. Combining this with (1.8), we obtain (1.6). $\Box$

As a corollary, we obtain the following

Lemma 2. *Let u, h be as in the previous lemma, and assume that $F \geq 0$. Then for $0 \leq t \leq t_0$,*

$$\sup_{0<s<t} \int_{D_s} e(u)\,dx \;\leq C \int_{D_0} e(u)\;dx \;+\; Ct \int_{K_{0,t}} h^2\;dx\,dt \tag{1.9}$$

$$\leq C \int_{D_0} e(u)\;dx \;+\; Ct^2 \sup_{0<s<t} \int_{D_s} h(x,s)^2\;dx \tag{1.10}$$

Proof. Note from Cauchy's inequality that $u_t h \leq \frac{\epsilon}{2} u_t^2 + \frac{1}{2\epsilon} h^2 \leq e(u) + \frac{1}{2\epsilon} h^2$, where we have used the fact that $F \geq 0$.

Again noting that $F \geq 0$, it follows from (1.6) that

$$\begin{aligned} \int_{D_t} e(u)\;dx \;&\leq \int_{D_0} e(u)\;dx \;+ \int_{K_{0,t}} u_t h\;dx\,dt \\ &\leq \int_{D_0} e(u)\;dx \;+\epsilon \int_{K_{0,t}} e(u)\,dx\,dt \;\; + \frac{1}{2\epsilon} \int_{K_{0,t}} h^2\;dx\,dt \\ &= \int_{D_0} e(u)\;dx \;+\epsilon \int_0^t \int_{D_s} e(u)\,dx\,ds \;\; + \frac{1}{2\epsilon} \int_{K_{0,t}} h^2\;dx\,dt \\ &\leq \int_{D_0} e(u)\;dx \;+\epsilon t \left(\sup_{0<s<t} \int_{D_s} e(u)\,dx \right) \;\; + \frac{1}{2\epsilon} \int_{K_{0,t}} h^2\;dx\,dt \end{aligned}$$

It follows that in fact

$$\sup_{0<s<t} \int_{D_s} e(u)\,dx \;\leq \int_{D_0} e(u)\;dx \;+\epsilon t \left(\sup_{0<s<t} \int_{D_s} e(u)\,dx \right) \;\; + \frac{1}{2\epsilon} \int_{K_{0,t}} h^2\;dx\,dt$$

We choose $\epsilon = \frac{1}{2t}$ to arrive at the (1.9), and (1.10) follows directly from (1.9). $\square$

We finish this section by noting

Lemma 3. *If $F = 0$ (ie, if we consider the linear wave equation) then the above conclusions remain true if we only assume that $h \in L^2_{loc}([0,T] \times \mathbb{R}^n))$ and the initial data u_0, u_1 belong to $H^1_{loc} \times L^2_{loc}(\mathbb{R}^n)$.*

The assumptions of this lemma are essentially the assumptions under which we established existence of solutions (for somewhat more general hyperbolic equations).

Proof. Let $u_0^\epsilon, u_1^\epsilon$ and h_ϵ be sequence of smooth functions converging to u_0, u_1 and h in $H^1_{loc}(\mathbb{R}^n)$, $L^2_{loc}(\mathbb{R}^n)$ and $L^2_{loc}([0,T] \times \mathbb{R}^n)$ respectively, and let u_ϵ denote the solution of the problem

$$\Box u_\epsilon = h_\epsilon \text{ in } [0,T] \times \mathbb{R}^n, \qquad (u_\epsilon, u_{\epsilon\, t})|_{t-0} = (u_0^\epsilon, u_1^\epsilon).$$

Fix $\epsilon, \delta > 0$ and let $w := u_\epsilon - u_\delta$. Then w solves

$$\Box w = h_\epsilon - h_\delta \text{ in } [0,T] \times \mathbb{R}^n, \qquad (w, w_t)|_{t=0} = (u_0^\epsilon - u_1^\delta, u_0^\epsilon - u_1^\delta).$$

Since the initial data and the right-hand side are smooth, the previous two lemmas apply to w (with $f = 0$ of course). Thus for example

$$\sup_{0<s<t} \int_{D_s} \frac{1}{2}(w_t^2 + |Dw|^2)\,dx \ \leq C \int_{D_0} (u_0^\epsilon - u_1^\delta)^2 + |D(u_0^\epsilon - u_1^\delta)|^2 \ dx$$
$$+ \ Ct \int_{K_{0,t}} (h_\epsilon - h_\delta)^2 \ dx\,dt.$$

The right-hand side tends to zero as $\epsilon, \delta \to 0$, so the same is true of the left-hand side. Notice also that for every t,

$$\begin{aligned} \int_{D_t} |w|^2 \ dx \ &= \int_{D_t} \left| w(0,x) + \int_0^t w_t(s,x) ds \right|^2 dx \\ &\leq C \int_{D_t} |w(0,x)|^2\,dx + Ct \int_{D_t} \int_0^t |w_t(s,x)|^2 ds dx \end{aligned} \tag{1.11}$$

We have already shown that the right-hand side tends to 0 as $\epsilon, \delta \to 0$. By combining this with the previous estimate, we find that

$$\sup_{0<s<t} \left(\|w_t\|^2_{L^2(D_t)} + \|w\|^2_{H^1(D_t)} \right) \to 0 \quad \text{ as } \epsilon, \delta \to 0.$$

And this implies that the sequence $\{u_\epsilon\}$ is Cauchy $C([0,T];H^1)$ and that $\{u_{\epsilon\, t}\}$ is Cauchy $C([0,T];L^)$. Since these $\{u_\epsilon\}_{\epsilon>0}$ and $\{u_{\epsilon\, t}\}_{\epsilon>0}$ clearly converge to u and u_t, respectively, in suitable weak topologies as $\epsilon \to 0$, it follows that they converge to the same limits in $C([0,T];H^1(D_t))$ and $C([0,T];L^2(D_t))$ respectively.

This convergence is strong enough that we can deduce that u satisfies (1.6) and (1.9) by noting that these conclusions hold for every $\epsilon > 0$, and then passing to limits. □

2. well-posedness for Lipschitz nonlinearities

We now use the above estimates to prove well-posedness of solutions for the semilinear wave equation (0.1) when f is Lipschitz. Altogether, our results show that in this siuation, the initial value problem (0.1), (0.2) has a unique solution ,for $(u_0, u_1) \in H^1_{loc} \times L^2_{loc}$ (Theorem 2.1), that the solutions depend continuously on the initial data (Lemma 4), that the solution enjoys better regularity properties when $(u_0, u_1) \in H^2_{loc} \times H^1_{loc}$ (Lemma 5), and that if the associated energy functional has a positivity property, then the total energy is conserved (Lemma 6.)

W will take for granted results about the solvability of the linear wave equation, (1.5) with $f(u) = 0$, with initial data as in (0.2). In fact, explicit formulas for the solutions are known, expressed in terms of convolutions of the Cauchy data (u_0, u_1) and the right-hand side h with certain fundamental solutions.

Given a function $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, we will write $u(t)$ to denote the function $\mathbb{R}^n \to \mathbb{R}$ that we get when we fix the time variable to equal t, so that $(u(t))(x) = u(x,t)$.

As this notation suggests, it is often useful to view functions on $[0,T] \times \mathbb{R}^n$ as maps from the time interval $[0,T]$ into spaces of functions on $\mathbb{R}^n$. For example, if U is a subset of $\mathbb{R}^n$, then $u \in L^\infty([0,T];H^1(U))$ if

$$\big\| \|u(\cdot)\|_{H^1(D_t)} \big\|_{L^\infty([0,T])} = \text{ess sup}_{t\in[0,T]} \left(\int_U u^2(x,t) + |\nabla u(x,t)|^2 \ dx \right)^{1/2} < \infty.$$

In addition, we write $u \in L^\infty([0,T]; H^1_{loc}(\mathbb{R}^n))$ if $u \in L^\infty([0,T]; H^1(K))$ for every compact $K \subset \mathbb{R}^n$. And we also write for example $u \in L^\infty([0,T]; H^1(D_t))$ if $\operatorname{ess\,sup}_{t\in[0,T]} \|u(t)\|_{H^1(D_t)} < \infty$, although this is perhaps a mild abuse of notation.

Theorem 2.1. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is C^1 and that there exists some constant C such that $|f'(z)| \le C$ for all $z \in \mathbb{R}$. Assume that $(u_0, u_1) \in H^1_{loc} \times L^2_{loc}(\mathbb{R}^n)$. Then for every $T > 0$, there exists a unique solution $u : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ of* (0.1), (0.2) *such that $u \in L^\infty([0,T]; H^1_{loc}(\mathbb{R}^n))$ and $u_t \in L^\infty([0,T]; L^2_{loc}(\mathbb{R}^n))$*

Proof. 1. We first solve the equation on a cone K with vertex (x_0, t_0), where t_0 will be fixed later. We will do this using a fixed point argument. We define the function space

$$X := \{u \in L^2(K) : u \in C([0,t_0]; H^1(D_t)), u_t \in C([0,t_0]; L^2(D_t))\}$$

We define the norm

$$\|u\|_X^2 := \sup_{0 \le t \le t_0} \int_{D_t} (u^2 + u_t^2 + |\nabla u|^2)\, dx. \tag{2.1}$$

This norm makes X into a Banach space.

We define a nonlinear mapping $L : X \to X$ by specifying that $L(v) = u$ if u

$$\Box u = -f(v) \qquad \text{in } K, \tag{2.2}$$

with the Cauchy data

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x) \qquad \text{for } x \in D_0.$$

It is a standard fact about linear wave equations that this problem has a unique solution.

2. We first verify that L maps X into X as claimed.

2a. It follows from Lemma 2 (with $F(u) = 0$, since (2.2) is a linear equation for u, once $v \in X$ is fixed) that for $u = L(v)$,

$$\int_{D_t} \frac{1}{2}(u_t^2 + |\nabla u|^2)\, dx \le C \int_{D_0} \frac{1}{2}(u_1^2 + |\nabla u_0|^2)\, dx + Ct^2 \sup_{t\in[0,t_0]} \|f(v)\|^2_{L^2(D_t)}.$$

The right-hand side is finite by the assumptions on the initial data, and because $|f(v)| \le C(|v| + 1)$ and $v \in X$.

2b. To prove that $u \in X$, we must still show that $\sup_t \|u\|_{L^2(D_t)}$ is bounded. For every t, we argue as in (1.11) to find that

$$\begin{aligned} \int_{D_t} u(x,t)^2\, dx \ &\le\ C \int_{D_t} t \int_0^t u_t^2(x,s)\, ds dx \ +\ \|u_0\|^2_{L^2(D_0)} \\ &\le\ Ct_0^2 \sup_{s\in[0,t_0]} \int_{D_s} u_t^2(s,x) dx + \|u_0\|^2_{L^2(d_0)}. \end{aligned} \tag{2.3}$$

Since we have already bounded $\sup_{s\in[0,t_0]} \int_{D_s} u_t^2(s,x)dx$, this shows that $u \in X$.

3. We next claim that if t_0 is chosen to be sufficiently small, then L is a contraction mapping. Indeed, let $u_1 = L(v_1), u_2 = L(v_2)$ for $v_1, v_2 \in X$. Then $w := u_1 - u_2$ solves

$$\Box w = f(v_2) - f(v_1) \qquad \text{in } K,$$

with initial data $w = w_t = 0$ on D_0. Again using Lemma 2, we see that,

$$\int_{D_t} \frac{1}{2}(w_t^2 + |\nabla w|^2)\, dx \le Ct_0^2 \sup_{s\in[0,t_0]} \|f(v_1) - f(v_2)\|^2_{L^2(D_s)} \tag{2.4}$$

for every $t \in [0, t_0]$ And since f is Lipschitz,

$$\sup_t \|f(v_1) - f(v_2)\|^2_{L^2(D_t)} \leq C \sup_t \|v_1 - v_2\|^2_{L^2(D_t)} \leq C\|v_1 - v_2\|^2_X.$$

Also, arguing as in (2.3) and using the fact that $w = 0$ at time 0,

$$\int_{D_t} w(x,t)^2 \, dx \ \leq \ Ct_0^2 \sup_{s\in[0,t_0]} \int_{D_s} w_t^2(s,x)dx \tag{2.5}$$

Combining (2.4) and (2.5), we obtain (recalling that $w = L(v_1) - L(v_2)$)

$$\|L(v_1) - L(v_2)\|^2_X \leq C(t_0^2 + t_0^4)\|v_1 - v_2\|^2_X \tag{2.6}$$

By choosing $t_0 > 0$ sufficiently small (depending only on the nonlinearity f, we can arrange that $C(t_0^2 + t_0^4) < 1$, and so for this choice of t_0, $L : X \to X$ is a contraction mapping. Note that the choice of t_0 is independent of the intial data (u_0, u_1).

4. It follows from the contraction mapping principle (also known as Banach's fixed point theorem) that L has a unique fixed point. That is, there exists a unique $u \in X$ such that $L(u) = u$. This says exactly that u solves the equation (0.1) in K and satisfies the initial condition (0.2) on D_0.

Note also that the analog of L is still a contraction mapping on any cone $K' = \{(x,t) \ : \ |x - x'| < t' - t\}$ if $0 < t' < t_0$, and hence there is a unique solution of the initial value problem for (0.1) on every such cone.

5. Now we can cover $\mathbb{R}^n \times [0, \frac{t_0}{2}]$ by a countable collection of cones $\{K_i\}$, all with height t_0. We can do this in such a way that any compact set intersects only finitely many cones. We let $u_i(x,t)$ denote the solution of (0.1) on K_i, which we have shown above to exist. We then define

$$u(x,t) = u_i(x,t) \quad \text{ for } (x,t) \in K_i.$$

To see that this is well-defined, suppose that $(\bar{x}, \bar{t}) \in K_i \cap K_j$. For $\epsilon > 0$ let $\bar{t}_\epsilon := \bar{t} + \epsilon$, and note that for sufficiently small $\epsilon > 0$, $K' = \{(x,t) : |x - \bar{x}| < \bar{t}_\epsilon - t + \epsilon\}$ is contained in $K_i \cap K_j$. Thus $u_i = u_j$ on K', by uniqueness. Since $(\bar{x}, \bar{t})$ was an arbitrary point in $K_i \cap K_j$, it follows that $u_i = u_j$ throughout $K_i \cap K_j$. Thus u is well-defined.

6. Note that any compact set $V \times t_0/2 \subset \mathbb{R}^n \times [0, \infty)$ is contained in a finite subcollection of cones K_i. It thus follows that at time $t_0/2$, $u(t) \in H^1_{loc}$ and $u_t \in L^2_{loc}(\mathbb{R}^n)$. We can therefore repeat the above argument to find a unique solution on $\mathbb{R}^n \times [\frac{1}{2}t_0, t_0]$, By iterating this procedure, the solution can be extended to arbitrarily large times.

$\square$

Lemma 4. *Assume that f be as in the previous theorem. Let K be any cone of the form* (1.4). *Then there exists a constant $C = C(K,f)$ with the following property:*

Let u and $\tilde{u}$ solve (0.1) *with initial data (u_0, u_1) and $(\tilde{u}_0, \tilde{u}_1)$ respectively. Define the norm $\|\cdot\|_X$ as in* (2.1). *Then*

$$\|u - \tilde{u}\|_X \leq C(\|u_0 - \tilde{u}_0\|_{H^1(D_0)} + \|u_1 - \tilde{u}_1\|_{L^2(D_0)}).$$

Proof. **1**. We first consider a cone K with vertex (t_0, x_0) such that t_0 is sufficiently small. We will later remove this smallness condition. Note that $w := u - \tilde{u}$ solves

$$\Box w = f(\tilde{u}) - f(u) \quad \text{ in } K, \qquad (w, w_t)|_{t=0} = (u_0 - \tilde{u}_0, u_1 - \tilde{u}_1).$$

First, as in (2.3) we have

$$\sup_{s\in[0,t_0]}\int_{D_s} w(s,x)^2\ dx \ \le\ Ct_0^2 \sup_{s\in[0,t_0]}\int_{D_s} w_t^2(s,x)dx + \|u_0-\tilde u_0\|^2_{L^2(d_0)}. \tag{2.7}$$

Thus energy estimates for the linear wave equation imply that

$$\begin{aligned}\sup_{0<t<t_0}\int w_t^2+|Dw|^2\ dx \le C\int_{D_0} |D((u_0-\tilde u_0)|^2 + (u_1-\tilde u_1)^2\ dx \\ + Ct_0^2\sup_{0<t<t_0}\int_{D_t}(f(\tilde u)-f(u))^2\ dx \\ \le C\int_{D_0} |D((u_0-\tilde u_0)|^2 + (u_1-\tilde u_1)^2\ dx \\ + Ct_0^2\sup_{0<t<t_0}\int_{D_t} w^2\ dx\end{aligned}$$

since $|f(u)-f(\tilde u)| \le C|u-\tilde u| = C|w|$. Combining this with (2.7), taking t_0 sufficiently small, and rearranging, we arrive at the conclusion of the lemma.

2. Now consider a cone $K = K(t_*,x_*)$ with vertex (t_*,x_*) for arbitrary t_*. We claim that the conclusion in this case follows by covering the possibly large cone $K(t_*,x_*)$ by smaller cones and using Step 1. To see this, let $K_{0,t_0/2} := \{(t,x)\in K : 0<t\le t_0/2\}$, where t_0 is the number chosen in Step 1. Then we can cover $K_{0,t_0/2}$ with finitely many cones $K_1,\dots K_m$ of height t_0 and centered at points $x_1,\dots,x_m$. Applying the above lemma on each smaller cone, we find that (using notation that should be self-explanatory)

$$\begin{aligned}&\sup_{0<t<t_0/2}(\|u(t)\|^2_{H^1(D_t(t_*,x_*))}+\|u_t(t)\|^2_{L^2(D_t(t_*,x_*))})\\ &\qquad\le\sum_{i=1}^m\sup_{0<t<t_0/2}\Big(\|u(t)\|^2_{H^1(D_t(t_0,x_i))}+\|u_t(t)\|^2_{L^2(D_t(t_*,x_*))}\Big)\\ &\qquad\le\sum_{i=1}^m\|u\|^2_{X(K(t_0,x_i))}\\ &\qquad\le C\sum_{i=1}^m\Big(\|u_0\|^2_{H^1(D_t(t_0,x_i))}+\|u_1(t)\|^2_{L^2(D_t(t_0,x_i))}\Big)\\ &\qquad\le C(\|u_0\|^2_{H^1(D_0(t_*,x_*))}+\|u_1(t)\|^2_{L^2(D_0(t_*,x_))}\end{aligned}$$

Repeating this argument for times $t\in t_0/2,t_0$ etc, we eventually cover the whole cone. ☐

Lemma 5. *Let f be as in the previous results. If $(u_0,u_1)\in H^2_{loc}\times H^1_{loc}$, then the solution u of* (0.1), (0.2) *proved to exist in* (2.1) *satisfies*

$$u\in H^2_{loc}(\mathbb{R}\times\mathbb{R}^n).$$

Proof. It suffices to show $u\in H^2(K)$ for any cone K. We henceforth fix such a cone.

For any $k\in 1,\dots,n$, let $\tau_k^h u(x) := u(x+he_k)$. In this notation the difference quotient $D_k^h u$ can be written $D_k^h u = \frac{1}{h}(\tau_k^h u - u)$.

Note that $w := \tau_k^h u$ solves (0.1) with initial data $(w,w_t)|_{t=0} = (\tau_k^h u_0, \tau_k^h u_1)$.

So we can apply the previous lemma on the cone K to find that

$$\|D_k^h u\|_{X(K)} = \frac{1}{h}\|\tau_k^h u - u\|_{X(K)} \le \frac{C}{h}(\|\tau_k^h u_0 - u_0\|_{H^1(D_0)} + \|\tau_k^h u_1 - u_1\|_{L^2(D_0)}$$
$$C(\|D_k^h u_0\|_{H^1(D_0)} + \|D_k^h u_1\|_{L^2(D_0)}).$$

Basic results about difference quotients imply that the right-hand side is bounded by $C(\|u_0\|_{H^2(V)} + \|u_1\|_{L^2(V)}$ for some set V such that $D_0 \subset\subset V$, and all h sufficiently small. Recalling the definition of the X norm, this implies that there exists some C such that $\|D_k^h Du\|_{H^1(K)} + \|D_k^h u_t\|_{H^1(K)} \le C$ for all small h. Thus the basic result on difference quotients implies that we second derivatives $u_{x_i x_k}$ and u_{tk} exist for all $i, k = 1, \ldots, n$ and are bounded in $L^2(K)$ by the same constant C. Finally, the equation implies that[1] $u_{tt} = \Delta u - f(u)$, and since we have just shown that the right-hand side belongs to $L^2(K)$, we conclude that the same is true for u_{tt} as well. This completes that $u \in H^2(K)$. □

Lemma 6. *Assume that the assumptions of the previous results hold, and that $f = F'$ with $F \ge 0$. Assume also that $(u_0, u_1) \in H^1 \times L^2(\mathbb{R}^n)$ are initial data for* (0.1) *such that*

$$\int_{\mathbb{R}^n} \frac{1}{2}(|Du_0|^2 + u_t^2) + F(u)\ dx := E_0 < \infty.$$

Then if u solves (0.1) *with the given data, and if $e(u) := \frac{1}{2}(|Du|^2 + u_t^2) + F(u)$, the identity*

$$\int_{\mathbb{R}^n} e(u(t))\, dx = E_0 \quad \text{for all } t \tag{2.8}$$

holds.

Note that in (1.3) we gave a formal proof of this identity if the solution u is known to be smooth and compactly supported. Here H^2 (in both the x and t variables) is sufficient smoothness to justify the computations that lead to (1.3). This lemma shows that this result is valid for the solutions we have found above, although the solutions are not compactly supported in general and are not smooth enough to justify the computations (1.1), (1.2).

The assumption that $F \ge 0$ makes the proof easier but it unnecessarily strong and could be relaxed somewhat.

Proof. **1**. First let us assume that $(u_0, u_1) \in H^2_{loc} \times H^1_{locc}$, in addition to the assumptions above. Then it follows from Lemma 5 that u satisfies the hypotheses of Lemma 1. So for $t > 0$ fixed, we conclude from facts such as (1.8) (established during the proof of Lemma 1) and the fact that $F \ge 0$ that

$$\int_{D_t(T,0)} e(u(t))\ dx - \int_{D_0(T,0)} e(u(0))\ dx = \frac{1}{\sqrt{2}}\int_{R_{0,t}(T,0)} (-e(u) + \nu_x \cdot \nabla u u_t) d\sigma \le 0. \tag{2.9}$$

In particular $\int_{D_t(T,0)} e(u)\ dx \le E_0$ for every t, T. If we fix t and let T tend to ∞, we deduce that $\int_{\mathbb{R}^n} e(u(t))\ dx \le E_0$.

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[1] strictly speaking, at this point we should write the equation in a weak form that only requires one t derivative of u, and then find that this weak form of the equation implies that the weak $\frac{\partial}{\partial t}$ derivative of u_t exists and is given by $\Delta u - f(u)$.

Next note that

$$\int_t^\infty (\int_{R_{0,t}(T,0)} e(u)d\sigma)dT \le C\int_0^t \int_{\mathbb{R}^n} e(u)\ dx\,dt \le CtE_0. \tag{2.10}$$

The easiest way of proving this is by an appeal to the "coarea formula" (see below). Alternatively, one can also note that for every $t \le T$,

$$\int_{R_{0,t}(T,0)} e(u)d\sigma = \sqrt{2}\int_0^t \int_{\partial B_{T-t}(x_0)} e(u)d\sigma'\ dt$$

where σ denotes n-dimensional volume measure on a hypersurface in a $\mathbb{R}^{n+1}$, and σ' denotes $n-1$-dimensional volume measure on a hypersurface in $\mathbb{R}^n$. Then (2.10) follows by substituting this into the left-hand side and changing the order of integration (and also noting that the integrand is nonnegative).

It follows from (2.10) that we can find a sequence $T_k \to \infty$ such that

$$\int_{R_{0,t}(T_k,0)} e(u)d\sigma \to 0.$$

Since the right-hand side of (2.9) is easily seen to be bounded in absolute value by $2\int_{R_{0,t}(T_k,0)} e(u)d\sigma$, we find by writing (2.9) for every T_k and letting k tend to ∞ that

$$\int_{\mathbb{R}^n} e(u(t))\,dx = E_0$$

which proves the lemma when the initial data are smooth enough.

2. If it is only true that $(u_0,u_1) \in H^1 \times L^2$, then one can approximate this initial data by a sequence $\{(u_0^\epsilon, u_1^\epsilon) \in H^2_{loc} \times H^1_{\text{loc}})$ such that $(u_0^\epsilon, u_1^\epsilon) \to (u_0,u_1)$ in $H^1 \times L^2$. Let u^ϵ denote the solution of (0.1) with initial data $(u_0^\epsilon, u_1^\epsilon)$. Using Lemma 4 (continnuous dependence of the solution on the data) and the fact that the conclusions hold for u_ϵ, for every $\epsilon > 0$, one can argue that the conclusions are satisfied for u as well.

□

3. Nonlinearities with arbitrary growth

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Next we consider f that can grow arbitrarily quickly, but that satisfies a positivity condition. We will prove

Theorem 3.1. *Assume that f is locally Lipschitz and that $0 \le uf(u) \le CF(u)$ for all $u \in \mathbb{R}$, where $F(s) = \int_0^s f(t)\,dt$.*

Assume that $(u_0,u_1) \in H^1 \times L^2$ are such that

$$E_0 := \int_{\mathbb{R}^n} \frac{1}{2}(|Du_0|^2 + u_1^2) + F(u_0)\ dx < \infty.$$

Then there exists a function $u \in H^1_{loc}([0,\infty)\times\mathbb{R}^n)$ solving the equation $\Box u + f(u) = 0$ in the following weak sense:

$$\iint_{[0,\infty)\times\mathbb{R}^n} [-v_t u_t + Dv\cdot Du + f(u)v]\ dtdx = 0 \qquad \textit{for all } v \in C_c^\infty((0,\infty)\times\mathbb{R}^n).$$

and such that $u|_{t=0} = u_0$(in the trace sense). Moreover,

$$E(u(t)) := \int_{\mathbb{R}^n} \frac{1}{2}(|Du(t)|^2 + u_t^2(t)) + F(u(t))\ dx \le E_0$$

for all $t \geq 0$.

Proof. **1**. Let

$$f_k(s) := \begin{cases} f(-k) & \text{if } s \leq -k \\ f(s) & \text{if } |s| \leq k \\ f(k) & \text{if } s \geq k. \end{cases}$$

Note that f_k is Lipschitz. so the equation $\Box u + f_k(u) = 0$ satisfies the hypotheses of Theorem 2.1 in the previous section. Let u_k solve

$$\Box u_k + f_k(u_k) = 0, \qquad (u_k, u_{k,t})|_{t-0} = (u_0, u_1).$$

Also, define $F_k(s) = \int_0^s f(t)\ dt$. Note that $F_k(s) = F(s)$ if $|s| \leq k$, and if $s > k$, then

$$F_k(s) = F(k) + (s-k)f(k) \geq C^{-1}kf(k) + (s-k)f(k) \geq C^{-1}sf_k(s) \geq 0 \tag{3.1}$$

for a constant that does not depend on s. (Similarly for $s < -k$). In particular, this implies that the hypotheses of Lemma 6 are satisfied, so that the energy identity (2.8) holds (with F replaced by F_k).

$\square$