In the final section of these notes we prove the following

Theorem 1. *Consider the equation*

$$u_t + Lu = f$$

where L is a divergence-form elliptic operator

$$Lu = \sum_{i,j=1}^{n} (a^{ij}(x,t)u_{x_i})_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} + cu$$

and $a^{ij} \in C^1(U_T)$, $b^i, c \in L^\infty(U_T)$. Also assume that $\sum_{i,j=1}^n (a^{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2$ at a.e. $(x,t) \in U_T$.

Assume that $\mathbf{u} \in L^2(0,T;H^1(U))$ with $\mathbf{u}' \in L^2(0,T;H^{-1}(U))$ and that for a.e. $t \in [0,T]$, the identity

$$\langle \mathbf{u}', v\rangle + B[\mathbf{u}, v; t] = (\mathbf{f}, v) \tag{1}$$

is valid for every $v \in H_0^1(U)$, using notation (such as the bilinear form B) from Evans.

Then for any $V \subset\subset U$ and $T_0 \in (0,T)$, the restriction of $\mathbf{u}$ to $V \times [T_0,T]$ belongs to $L^\infty(T_0,T;H^1(V)) \cap L^2(T_0,T;H^2(V))$, and the restriction of $\mathbf{u}'$ belongs to $L^2(T_0,T;L^2(V))$. Moreover, the following estimates hold:

$$\boxed{\sup_{T_0\leq t\leq T} \int_V |Du|^2\ dx + \iint_{V\times[T_0,T]} |D^2u|^2\, dx\, dt \ \leq\ C\left(\iint_{U_T} u^2\ dx\ dt + \iint_{U_T} f^2 dx\, dt\right)}$$

and

$$\boxed{\iint_{V\times[T_0,T]} |u_t|^2\, dx\, dt \ \leq\ C\left(\iint_{U_T} u^2\ dx\ dt + \iint_{U_T} f^2 dx\, dt\right).}$$

The constants depend on V, T_0, the norms of the coefficients, the parabolicity constant θ

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We start with a lengthy discussion in which we prove estimates for smooth solutions of the heat equation. These provide a model for the harder estimates of Theorem 1 .

1. Estimates for the inhomogeneous heat equation

Assume that u soves

$$u_t - \Delta u = f \tag{2}$$

in $U_T := U \times [0,T]$, where U is a bounded open subset of $\mathbb{R}^n$ with smooth boundary. We assume that u is smooth enough to justify the calculations that follow. (We will eventually discuss how to carry out these calculations rigorously for solutions that are only know at the outset to be of rather low regularity.)

1.1. basic estimates.

Multiply (2) by u and rewrite to find that

$$\boxed{\frac{d}{dt}(\frac{1}{2}u^2) + |Du|^2 = \operatorname{div}(uDu) + fu.} \tag{3}$$

Multiply by u_t and rewrite to find that

$$\boxed{u_t^2 + \frac{d}{dt}(\frac{1}{2}|Du|^2) = \operatorname{div}(u_tDu) + fu_t.} \tag{4}$$

Multiply by $-\Delta u$ and rewrite to find that

$$\boxed{\frac{d}{dt}(\frac{1}{2}|Du|^2) + (\Delta u)^2 = \mathrm{div}(u_t Du) - f\Delta u.} \tag{5}$$

If f is smooth enough, one can differentiate the equation and repeat the above sorts of computations.

1.2. **boundary regularity.** If u satisfies the boundary condition $u = 0$ on $\partial U \times [0,T]$, (ie, $\mathbf{u}(t) \in H^1_0(U)$ for a.e. t, in the notation from Evans' book) then formally $u_t \equiv 0$ on $\partial U \times [0,T]$, and so

$$\int_U \mathrm{div}(uDu)\,dx = \int_{\partial U} u\nu \cdot Du\,ds = 0, \qquad \int_U \mathrm{div}(u_t Du)\,dx = \int_{\partial U} u_t\nu \cdot Du\,ds = 0$$

for every time t. So one can integrate the above identities, and the boundary terms vanish. For example, from (3) we deduce that

$$\frac{d}{dt}\int \frac{1}{2}u^2\,dx \;+\; \int |Du|^2\,dx \le \frac{1}{2}\int f^2\,dx + \frac{1}{2}\int u^2\;dx.$$

And by Lemma 1 (see below; essentially Gronwall's inequality plus another easy argument) it follows that

$$\boxed{\sup_{0<t\le T}\int u^2\,dx \;+\int_0^T\int |Du|^2\,dxdt \;\;\le\;\; C\left(\int u^2 dx\Big|_{t=0} + \int_0^T\int f^2\,dx\,dt\right).} \tag{6}$$

Similarly, by integrating (4) and using Cauchy's inequality we obtain

$$\frac{1}{2}\int u_t^2\,dx + \frac{d}{dt}\int \frac{1}{2}|Du|^2 dx \le \frac{1}{2}\int f^2 dx.$$

Then Lemma 1 yields

$$\boxed{\sup_{0<t\le T}\int |Du|^2\,dx \;+\int_0^T\int u_t^2\,dxdt \;\;\le\;\; C\left(\int |Du|^2 dx\Big|_{t=0} + \int_0^T\int f^2\,dx\,dt\right).} \tag{7}$$

Similarly, from (5) we deduce that

$$\boxed{\sup_{0<t\le T}\int |Du|^2\,dx \;+\int_0^T\int \Delta u^2\,dxdt \;\;\le\;\; C\left(\int |Du|^2 dx\Big|_{t=0} + \int_0^T\int f^2\,dx\,dt\right).} \tag{8}$$

The only new part in (8), as compared to (7), is the estimate of $\|\Delta u\|_{L^2(U_T)}$, which we could also have obtained from (7) and the fact that $\Delta u = u_t - f$. Or conversely, we could have obtained the estimate of u_t from the estimate of Δu. (We will do this later.)

Since $u(\cdot) \in H^1_0$, elliptic regularity theorems imply that $\|u(\cdot,t)\|_{H^2(U)} \le C\|\Delta u(\cdot,t)\|_{L^2}$ for every t, so (8) provides control over $\|u\|_{L^2_t H^2_x}$.

1.3. **interior regularity.** In deriving (6), (7), (8), we assumed that $u = 0$ on $\partial U \times [0,T]$, and these estimates are vacuous unless we have control over the norms of $u(x,0)$ that appear on the right-hand sides. We can still get interior regularity, however, even if we know nothing about the the initial or boundary data. For this, fix $V \subset\subset U$ and $T_0 \in (0,T)$, and let ζ be a smooth function such that $0 \le \zeta \le 1$ and

$$\zeta = 1 \text{ on } V \times [T_0,T], \qquad \zeta = 0 \text{ in a neighborhood of } (U \times \{t=0\}) \cup (\partial U \times [0,T]). \tag{9}$$

Basic interior regularity estimates will follow from multiplying (3), (4), (5) by ζ^2, integrating, and using Lemma 1.

First, multiplying (3) by ζ^2 and integrating in the x variables for fixed t, we find that

$$\frac{d}{dt}\int \frac{1}{2}\zeta^2 u^2\ dx + \int \zeta^2 |Du|^2 = -\int \zeta\zeta_t u^2 + 2\int u\ \zeta\ D\zeta\cdot Du + \int \zeta^2 f u$$

After rearranging and using some elementary inequalities, we find that

$$\frac{d}{dt}\int \frac{1}{2}\zeta^2 u^2\ dx + \int \zeta^2 |Du|^2 \le c_1 \int \zeta^2 u^2\ dx\ +\ c_2 \int \left[(\zeta_t^2 + |D\zeta|^2)u^2 + \zeta^2 f^2\right] \tag{10}$$

for every t. We use Lemma 1 to find that

$$\sup_{0\le t\le T}\int \frac{1}{2}\zeta^2 u^2\ dx + \int_0^T\int \zeta^2|Du|^2\,dx\,dt \le C\int_0^T\int \left[(\zeta_t^2 + |D\zeta|^2)u^2 + \zeta^2 f^2\right]\ dx\ dt.$$

where we have used the fact that $\zeta = 0$ at $t = 0$. Recalling the definition of ζ, we conclude that

$$\boxed{\sup_{T_0\le t\le T}\int_V u^2\ dx + \iint_{V\times[T_0,T]} |Du|^2\,dx\,dt\ \le\ C\left(\iint_{U_T\setminus(V\times[T_0,T])} u^2\ dx\ dt + \iint_{U_T} f^2 dx\,dt\right)} \tag{11}$$

The constant depends upon derivatives of ζ, among other things. Note that the support of $|\zeta_t|+|D\zeta|$ is contained in $U_T\setminus(V\times[T_0,T])$.

Similarly, we can repeat the arguments that lead from (4) to (7) and from (5) to (8), using the cutoff function instead of the boundary conditions. Thus, multiplying (5) by ζ^2 and more or less repeating the above calculations leads to

$$\sup_{0\le t\le T}\int \zeta^2|Du|^2\ dx + \int_0^T\int \zeta^2(\Delta u)^2\,dx\,dt \le C\int_0^T\int \left[(\zeta_t^2 + |D\zeta|^2)|Du|^2 + \zeta^2 f^2\right]\ dx\ dt. \tag{12}$$

Before putting a box around this equation, we improve it by showing that one can essentially replace Du by u on the right-hand side. To do this, fix $V\subset\subset W\subset\subset U$ and $0 < t_0 < T_0$. We can require that the test function ζ in the above inequality is supported in $W\times[t_0,T]$, and as before that $\zeta\equiv 1$ on $V\times[T_0,T]$. Then we can apply (11) with $V\times[T_0,T]$ replaced by $W\times[t_0,T]$ to find that

$$\begin{aligned}\int_0^T\int (\zeta_t^2 + |D\zeta|^2)|Du|^2\ dx\ dt &\le C\iint_{W\times[t_0,T]} |Du|^2\ dx\ dt\\ &\le C\left(\iint_{U_T\setminus(W\times[t_0,T])} u^2\ dx\ dt + \iint_{U_T} f^2 dx\,dt\right)\end{aligned}$$

Combining this with (12) we deuce that

(13)

$$\boxed{\sup_{T_0\le t\le T}\int_V |Du|^2\ dx + \iint_{V\times[T_0,T]} (\Delta u)^2\,dx\,dt\ \le\ C\left(\iint_{U_T\setminus(V\times[T_0,T])} u^2\ dx\ dt + \iint_{U_T} f^2 dx\,dt\right)}$$

And by either repeating these arguments with (5) as the starting point, or by using the equation $u_t = \Delta u - f$, we find that

$$\boxed{\iint_{V\times[T_0,T]} u_t^2 dx\,dt \le C\left(\iint_{U_T\setminus(V\times[T_0,T])} u^2\ dx\ dt + \iint_{U_T} f^2 dx\,dt\right)} \tag{14}$$

Finally, if we wish, we can apply interior *elliptic* regularity to further improve (13). This would ultimately lead to estimates of $\|u\|_{L^2(T_0,T;H^2(V_1))}$ for some $V_1\subset\subset V$

1.4. **Higher regularity.** If for example $f \in H^1(U_T)$, then we can differentiate the equation to find that u_{x_i} and u_t both satisfy equations of the form

$$\tilde{u}_t - \Delta \tilde{u} = \tilde{f}$$

where $\tilde{f} = f_t$ when $\tilde{u} = u_t$, and $\tilde{f} = f_{x_i}$ when $\tilde{u} = u_{x_i}$. Then we can obtain further regularity by applying the above estimates to $\tilde{u} = u_t$ or u_{x_i}. Similarly, if f has more derivatives, then we can differentiate the equation more times and obtain still higher regularity for u.

2. A USEFUL LEMMA

We have used the following lemma several times already in the above discussion.

Lemma 1. *Suppose that f, g, h are nonnegative functions on $[0, T]$, that f is absolutely continuous, and h is integrable, and that*

$$f' + g \leq c_1 f + c_2 h$$

at a.e. $t \in [0, T]$. Then given $T > 0$, there exists a constant C (depending on T, c_1, c_2) such that

$$\sup_{0<t\leq T} f(t) \; + \int_0^T g(t)dt \quad \leq \quad C\left(f(0) + \int_0^T h(t)\,dt\right). \tag{15}$$

The hypotheses clearly imply that g is integrable, since $0 \leq g \leq f' + c_1 f + c_2 h$.

Proof. The hypotheses imply that $f' \leq c_1 f + c_2 g$, and hence that $(e^{-c_1 t} f)' \leq c_2 e^{-c_1 t} h$ a.e.. By integrating and rearranging we deduce that

$$f(t) \leq e^{c_1 t} f(0) + c_2 \int_0^t e^{c_1(t-s)} h(s)\,ds. \tag{16}$$

for every $t \in (0, T]$. (This is just Gronwall's inequality). Now we integrate the hypotheses again to find that

$$f(T) + \int_0^T g(s)\;ds \leq f(0) + c_1 \int_0^T f(s)\,ds + c_2 \int_0^T gh(s)\,ds.$$

Using (16), this yields

$$\int_0^T g(t)\,dt \quad \leq \quad C\left(f(0) + \int_0^T h(t)\,dt\right).$$

where the constant C depends on T, c_1, c_2. □

3. PROOF OF THEOREM 1

This section is devoted to the proof of Theorem 1. Thus we consider the equation

$$u_t + Lu = f$$

where L is a divergence-form elliptic operator

$$Lu = \sum_{i,j=1}^n (a^{ij}(x,t) u_{x_i})_{x_j} + \sum_{i=1}^n b^i u_{x_i} + cu$$

and $a^{ij} \in C^1(U_T)$, $b^i, c \in L^\infty(U_T)$. We also assume that $\sum_{i,j=1}^n (a^{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2$ at a.e. $(x,t) \in U_T$.

In this discussion, unlike the above, we assume only that we have a weak solution and we do not make any *ad hoc* smoothness assumptions. We will only consider interior estimates. (Boundary regularity is discussed in Evans.)

(If we wanted to establish about higher regularity, we would assume more smoothness of the coefficients, since in order to establish higher regularity we would have to differentiate the equation.)

Precisely, we assume that $\mathbf{u} \in L^2(0,t;H^1_{\rm loc}(U))$ with $\mathbf{u}' \in L^2(0,T;H^{-1}(U))$ and that for a.e. $t \in [0,T]$, the identity

$$\langle \mathbf{u}', v\rangle + B[\mathbf{u}, v; t] = (\mathbf{f}, v) \tag{17}$$

is valid for every $v \in H^1_0(V)$, for every $W \subset\subset U$, using notation from Evans.

3.1. **interior $L^\infty L^2 \cap L^2 H^1$ estimates.** We first prove the analog of the interior regularity estimate (11). For these we do not need a^{ij} to be differentiable; L^∞ and uniform parabolicity are sufffficient.

For this, we fix ζ as in (9), and such that $\zeta \in C^\infty_c(U_T)$, where the subscript $_c$ indicates that ζ has compact support. We write $\boldsymbol{\zeta}(t) := \zeta(x,t)$. Then $v := \boldsymbol{\zeta}^2(t)\,\mathbf{u}(t)$ belongs to $H^1_0(U)$ hence is an acceptable "test function" in the definition (17) of a weak solution. Thus we find that

$$\langle \mathbf{u}', \boldsymbol{\zeta}^2\,\mathbf{u}\rangle + B[\mathbf{u}, \boldsymbol{\zeta}^2\,\mathbf{u}; t] = (\mathbf{f}, \boldsymbol{\zeta}^2\,\mathbf{u}) \tag{18}$$

for a.e. $t \in [0,T]$.

(To make things easier to read I am writing $\mathbf{u}$ instead of $\mathbf{u}(t)$ throughout, and similarly $\boldsymbol{\zeta}$. Also, in this discussion, whenever I integrate over x variables I write u and ζ instead of $\mathbf{u}, \boldsymbol{\zeta}$, since when we integrate in the $x$variables we are recalling that the abstract element say $\mathbf{u}(t)$ of a Hilbert space can be identified with a function $[\mathbf{u}(t)](x) = u(x,t)$, and we are performing operations like integration on the function $u(x,t)$.)

First, one can check (for example by approximating $\mathbf{u}$ by smooth functions) that, since ζ is a smooth function, $(\boldsymbol{\zeta}\mathbf{u})' \in L^2(0,T;H^{-1}(U))$, and that $(\boldsymbol{\zeta}\mathbf{u})' = \boldsymbol{\zeta}\mathbf{u}' + \boldsymbol{\zeta}'\mathbf{u}$. Thus

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\zeta}\mathbf{u}\|^2_{L^2} = \langle \boldsymbol{\zeta}\mathbf{u}', \boldsymbol{\zeta}\mathbf{u}\rangle + \langle \boldsymbol{\zeta}'\mathbf{u}, \boldsymbol{\zeta}\mathbf{u}\rangle \le \langle \mathbf{u}', \boldsymbol{\zeta}^2\mathbf{u}\rangle + \|\boldsymbol{\zeta}\mathbf{u}\|^2_{L^2} + \|\boldsymbol{\zeta}'\mathbf{u}\|^2_{L^2}. \tag{19}$$

Next, we write $B[u,v;t] := B_0[u,v;t] + B_1[u,v;t]$, where $B_0[u,v;t] := \int \sum_{i,j=1}^n a^{ij}(x,t) u_{x_i} v_{x_j}\,dx$, and $B_1[u,v;t] = B[u,v;t] - B_0[u,v;t]$. Then

$$B_0[\mathbf{u}, \boldsymbol{\zeta}^2\,\mathbf{u}; t] = \int \sum a^{ij} u_{x_i} (\zeta^2 u)_{x_j} dx = \int \sum a^{ij}\zeta^2 u_{x_i} u_{x_j}\,dx \;+\; 2\int \sum a^{ij} u_{x_i} u \zeta \zeta_{x_j}\,dx$$

where all the integrals are evaluated at time t. Using elementary inequalities and the parabolicity assumption, we deduce that

$$B_0[\mathbf{u}, \boldsymbol{\zeta}^2\,\mathbf{u}; t] \ge \frac{\theta}{2}\int \zeta^2 |Du|^2\,dx \;-C\int |D\zeta|^2 u^2. \tag{20}$$

Next, further elementary estimates show that

$$|B_1[\mathbf{u}, \boldsymbol{\zeta}^2\,\mathbf{u}; t]| \le \frac{\theta}{4}\int \zeta^2 |Du|^2 dx + C\int \zeta^2 u^2\,dx. \tag{21}$$

Finally,

$$(\mathbf{f}, \boldsymbol{\zeta}^2\,\mathbf{u}) \le \int \zeta^2 f^2 dx + \int \zeta^2 u^2\,dx \tag{22}$$

where again all the integrals are evaluated at time t.

Now we substitute (19), (20), (21), (22), into (18) and rearrange to find

$$\frac{d}{dt}\int \frac{1}{2}\zeta^2 u^2\ dx + \frac{\theta}{4}\int \zeta^2 |Du|^2 dx \le c_1 \int \zeta^2 u^2\ dx \ +\ c_2 \int \left[(\zeta_t^2 + |D\zeta|^2)u^2 + \zeta^2 f^2\right]$$

This is *exactly* the same as (10) (except for some different constants), which we found in a corresponding point of our formal discussion of the heat equation. Thus, exactly as before, we can use Lemma 1 and the definition of ζ to conclude that

$$\boxed{\sup_{T_0\le t\le T}\int_V u^2\ dx + \iint_{V\times[T_0,T]} |Du|^2\, dx\, dt \ \le\ C\left(\iint_{U_T\setminus(V\times[T_0,T])} u^2\ dx\ dt + \iint_{U_T} f^2 dx\, dt\right)} \tag{23}$$

3.2. **interior $L^\infty H^1 \cap L^2 H^2$ estimates.** Next we derive in this setting an estimate analogous to (13). We do this essentially by repeating the calculations by which we derived (13) from (5). However, we will simplify the work (and sweep most of the details under the carpet) by appealing to results we established when studying interior elliptic regularity.

So, we fix ζ as above, and we let $v := D_k^{-h}(\boldsymbol{\zeta}^2(t)\, D_k^h \mathbf{u}(t))$ for some k, where D_k^h denotes a difference quotient, following notation from Evans, Sections 5.8 and 6.3. As in the previous subsection, v is an acceptable "test function", and so from the definition (17) of a weak solution we find that

$$\langle \mathbf{u}', D_k^{-h}(\boldsymbol{\zeta}^2\, D_k^h\mathbf{u})\rangle = -B[\mathbf{u}, D_k^{-h}(\boldsymbol{\zeta}^2\, D_k^h\mathbf{u}); t] + (\mathbf{f}, D_k^{-h}(\boldsymbol{\zeta}^2\, D_k^h\mathbf{u})) \tag{24}$$

for a.e. $t\in[0,T]$. First note that, by integration by parts for difference quotients (see formula (16), section 6.3.1, Evans)

$$\langle \mathbf{u}', D_k^{-h}(\boldsymbol{\zeta}^2\, D_k^h\mathbf{u})\rangle = \langle D_k^h\mathbf{u}', \boldsymbol{\zeta}^2\, D_k^h\mathbf{u}\rangle = \langle \zeta D_k^h\mathbf{u}', \boldsymbol{\zeta}\, D_k^h\mathbf{u}\rangle.$$

So as in the previous subsection,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{\zeta} D_k^h\mathbf{u}\|_{L^2}^2 &= \langle \boldsymbol{\zeta} D_k^h\mathbf{u}', \boldsymbol{\zeta} D_k^h\mathbf{u}\rangle + \langle \boldsymbol{\zeta}' D_k^h\mathbf{u}, \boldsymbol{\zeta} D_k^h\mathbf{u}\rangle\\ &\le \langle \mathbf{u}', D_k^{-h}(\boldsymbol{\zeta}^2\, D_k^h\mathbf{u})\rangle + \|\boldsymbol{\zeta} D_k^h\mathbf{u}\|_{L^2}^2 + \|\boldsymbol{\zeta}' D_k^h\mathbf{u}\|_{L^2}^2.\end{aligned} \tag{25}$$

Next we consider the right-hand side of (24). We already looked carefully at expression of exactly this form when considering interior elliptic regularity. Indeed, the proof of Theorem 1 in Section 6.3.1 of Evans shows that

$$B[\mathbf{u}, D_k^{-h}(\boldsymbol{\zeta}^2\, D_k^h\mathbf{u}); t] - (\mathbf{f}, D_k^{-h}(\boldsymbol{\zeta}^2\, D_k^h\mathbf{u})) \ge \frac{\theta}{4}\int_U \zeta^2 |D_k^h Du|^2 - C\int_U (|Du|^2 + f^2 + u^2) dx$$

(This follows from combining equations (20) and (22) in section 6.3.1, recalling the definitions in (10)-(14) of the same section. This is where we use the assumption that a^{ij} is C^1; this was used in the arguments from Evans that we are quoting here.) Also, by assuming that ζ is supported in $W\times[t_0,T]$, where $t_0\in(0,T_0)$ and $W\subset\subset U$, we can replace U by W in the integrals on the right-hand side. By combining this with (24), (25) we deduce that

$$\frac{d}{dt}\|\boldsymbol{\zeta} D_k^h\mathbf{u}\|_{L^2}^2 + \int_U \zeta^2 |D_k^h Du|^2 \le C\int_W (|Du|^2 + f^2 + u^2) dx.$$

Note also that both sides of the equation vanish if $t < t_0$. Thus we deduce from Lemma 1 that

$$\sup_{0<t<T}\|\boldsymbol{\zeta} D_k^h\mathbf{u}\|_{L^2}^2 + \iint_{U_T} \zeta^2 |D_k^h Du|^2 dx\, dt \le C\int_{W\times[t_0,T]} (|Du|^2 + f^2 + u^2) dx\, dt.$$

Since this holds for every k and every sufficiently small h, the theorem on difference quotients implies that $u(\cdot,t)\in H^2(V)$ for every $t\in[T_0,T]$ and moreover that

$$\sup_{T_0<t<T}\|D\mathbf{u}(t)\|_{L^2(V)}^2 + \iint_{V\times[T_0,T]} |D^2u|^2 dx\, dt \le C\int_{W\times[t_0,T]} (|Du|^2 + f^2 + u^2) dx\, dt.$$

As in the proof of (13), we can replace the $|Du|^2$ term on the right-hand side by using the interior L^2H^1 estimates already established, which in this case means by appealing to (23) (with $V\times[T_0,T]$ replaced by $W\times[t_0,T]$). Thus we finally arrive at

$$\boxed{\sup_{T_0\le t\le T}\int_V |Du|^2\;dx + \iint_{V\times[T_0,T]} |D^2u|^2\,dx\,dt \;\le\; C\left(\iint_{U_T} u^2\;dx\;dt + \iint_{U_T} f^2 dx\,dt\right)} \tag{26}$$

In this case (unlike our earlier discussion of the heat equation) we have proved that this holds, starting only from the definition of a weak solution, and without making any unjustified regularity assumptions.

Finally, by using the equation $u_t = -Lu + f$ and (23), (26) we can argue that

$$\boxed{\iint_{V\times[T_0,T]} |u_t|^2\,dx\,dt \;\le\; C\left(\iint_{U_T} u^2\;dx\;dt + \iint_{U_T} f^2 dx\,dt\right).} \tag{27}$$

In order to justify this, first note that the estimates already obtained imply that $Lu(\cdot,t)\in L^2(V)$ for a.e. $t\in[T_0,T]$ (where this means Lu evaluated at tme t and restricted to V), and that $B[\mathbf{u}(t),v;t] = (Lu(\cdot,t),v)$ for every $v\in H^1_0(V)$. Thus the weak form of the equation implies that

$$\langle \mathbf{u}'(t), v\rangle = (-Lu(\cdot,t) + f(\cdot,t), v) \qquad \text{for every } v\in H^1_0(V) \text{ and a.e. time } t\in[T_0,T]$$

From this one can deduce that $\mathbf{u}'(t) = -Lu(\cdot,t)+f(\cdot,t)$, and then (27) follows from the estimates already proved.

Thus we have proved Theorem 1.