PDEII, possible questions for quiz # 2, spring 2008,

1. Suppose that U is a bounded open subset of $\mathbb{R}^n$, and suppose that $1 \le p < n$. Give an example of a sequence of functions $\{u_k\} \subset W^{1,p}(U)$ such that

$$\|u_k\|_{W^{1,p}(U)} \le C \qquad \text{but no subsequence converges to any limit in } L^{p^*}(U).$$

where p^* is the Sobolev exponent $p^* = \frac{np}{n-p}$.

(Hint: look for a sequence such that $u_k \to 0$ a.e. but $\|u_k\|_{L^{p^*}}$ is a constant independent of k.)

2. Give an example of a sequence of *radial* functions $u_k \in W^{1,p}(\mathbb{R}^n)$ for $1 \le p < n$ such that

$$\|u_k\|_{W^{1,p}(U)} \le C \qquad \text{but no subsequence converges to any limit in } L^p(\mathbb{R}^n).$$

(Similar hint to above.)

3. Let U be the unit ball in $\mathbb{R}^n$, and let p be a number such that $1 \le p < \infty$. Show by example that there exists a sequence of functions $\{u_k\} \subset W^{1,p}(U)$ such that

   (a) $\frac{\partial u_k}{\partial \nu} = 0$ on ∂U (in particular, each u_k should be smooth enough that $\frac{\partial u_k}{\partial \nu}$ is well-defined)

   (b) The sequence u_k converges in $W^{1,p}(U)$ to a limit u; and

   (c) $\frac{\partial u}{\partial \nu}$ is well-defined and does not identically vanish on ∂U.

This shows that on cannot hope to minimize any functional in a set of the form $\mathcal{A} := \{u \in H^1(U) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial U\}$. Indeed, we do not know how to make sense of $\frac{\partial u}{\partial \nu}$ for a general function $u \in H^1(U)$, since we have not established anything analogous to the trace theorem for $\frac{\partial u}{\partial \nu}$.

If I ask this question on a quiz, I will only ask you to write down the answer, but not to check in detail that the examples you give have the desired properties. (But you should write down a correct answer!)

4. Prove the following Poincaré type inequality:

If $U \subset \mathbb{R}^n$ is a bounded open set with smoooth boundary, then there exists a constant C (depending on the domain U) such that

$$\|u\|_{L^2(U)} \le C\left(\|Du\|_{L^2(U)} + \|Tu\|_{L^2(\partial U)}\right)$$

where $Tu \in L^2(\partial U)$ denotes the trace of u.

(This was needed in a previous homework problem. In your solution, feel free to write u instead of Tu for simplicity, when referring to the boundary values of u.)

5. Suppose that H and $\tilde{H}$ are Hilbert spaces, and that $A : H \to \tilde{H}$ is a bounded linear operator. Prove that if $u_k \rightharpoonup u$ weakly in H, then $Au_k \rightharpoonup Au$ weakly in $\tilde{H}$.

(This implies in particular that if $u_k \rightharpoonup u$ weakly in $H^1(U)$ then $Tu_k \rightharpoonup Tu$ weakly in $L^2(\partial U)$, where Tw denotes the trace of w. This also was needed for a previous homework problem. For this, you may need to recall the definition of the adjoint A^* of a bounded linear operator $A : H \to \tilde{H}$.)

6. Evans chapter 7 problem 1.

7. Evans chapter 7 problem 2.