Note: A standing homework assignment for students in MAT1501 is: let me know about any mistakes, misprints, ambiguities etc that you find in these notes.

1. MEASURES AND MEASURABLE SETS

If X is a set, let 2^X denote the power set of X, ie, the collection of all subsets of X.

A *measure* μ on X is defined to be a nonnegative, countably subadditive set function on X, ie, a function $\mu : 2^X \to [0,\infty]$ such that

$$\mu(\emptyset) = 0, \qquad \text{and } \mu(A) \le \sum \mu(A_i) \ \text{ whenever } A \subset \cup_i A_i \tag{1}$$

for any finite or countable collection of subsets $A_i \subset X$. Note in particular that the definition implies that $\mu(A) \le \mu(B)$ when $A \subset B$.

Remark 1. What we are calling a measure is called an *outer measure* in many books.

Given a measure μ on X, we say that a set $A \subset X$ is μ-measurable if

$$\mu(B) = \mu(B \cap A) + \mu(B \setminus A) \tag{2}$$

for all $B \subset X$. To establish that measurability of A, it is only necessary to check that $\mu(B) \ge \mu(B \cap A) + \mu(B \setminus A)$ for all B, since the opposite inequality follows from the definition of a measure. It is obvious from the definition that X and $\emptyset$ are measurable, and also that

$$A \text{ is } \mu\text{-measurable} \implies X \setminus A \quad \text{is } \mu\text{- measurable }. \tag{3}$$

Given a measure μ on X and a subset $B \subset X$, we define the function $\mu \llcorner B : 2^X \to [0,\infty]$by

$$\mu \llcorner B(A) = \mu(B \cap A).$$

It is straightforward to check that $\mu \llcorner B$ is a measure, regardless of whether or not$B$ is measurable.

The first theorem of measure theory is

Theorem 1. *Suppose that μ is a measure on a set X, and $\{A_i\}$ be a sequence of μ-measurable sets. Then*

i. *Both $\cup_{i=1}^\infty A_i$ and $\cap_{i=1}^\infty A_i$ are μ-measurable.*
ii. $\mu(\cup_{i=1}^\infty A_i) = \lim_{j\to\infty} \mu(\cup_{i=1}^j A_i)$.
iii. $\mu(\cap_{i=1}^\infty A_i) = \lim_{j\to\infty} \mu(\cap_{i=1}^j A_i)$*, as long as the right-hand side is finite.*
iv. *If the sets $\{A_i\}$ are pairwise disjoint, then $\mu(\cup_{i=1}^\infty A_i) = \sum_{i=1}^\infty \mu(A_i)$.*

Sometimes *(ii)* and *(iii)* are stated in the form

ii'. If $E_1 \subset E_2 \subset \ldots$ are measurable and $E_j \uparrow E$, then $\mu(E) = \lim_{j\to\infty} \mu(E_j)$.
iii'. If $E_1 \supset E_2 \supset \ldots$ are measurable and $E_j \downarrow E$, then $\mu(E) = \lim_{j\to\infty} \mu(E_j)$ as long as the right-hand side is finite.

sketch of proof. It is straightforward to verify by induction that conclusions (i) and (iv) hold for *finite* collections of sets. Once this is known, it follows that $\mu(\cup_{i=1}^\infty A_i) \ge \sum_{i=1}^j \mu(A_i)$ for every j, which in view of (1) implies that (iv) holds for a countable collection of pairwise disjoint sets as stated. Then (ii) can be deduced from (iv), and (iii) from (ii). To verify (i), it is easiest to note that for given $B \subset X$ (not necessarily measurable), the function $\mu \llcorner B : 2^X \to [0, \infty]$ defined by

$$\mu \llcorner B(A) = \mu(B \cap A)$$

is itself a measure; this is not hard to check. Moreover, every μ-measurable set is also $\mu \llcorner B$-measurable. Using these facts, (i) can be inferred from (ii) and (iii). $\square$

Conclusion (i) of the theorem, with (3), implies that the collection of μ-measurable sets is a σ-algebra, where we recall the definition:

A collection $S \subset 2^X$ is said to be a σ-algebra if $\emptyset, X \in S$,

$$A \in S \Longrightarrow X \setminus A \in S, \tag{4}$$

and

$$\text{if } A_1, A_2, \ldots \in S, \text{ then } \cup_{i=1}^{\infty} A_i \in S. \tag{5}$$

Clearly, if (4) and (5) hold, then so does

$$\text{if } A_1, A_2, \ldots \in S, \text{ then } \cap_{i=1}^{\infty} A_i \in S. \tag{6}$$

It is equally clear that (4) and (6) imply (5). It is not true in general that (5) and (6) imply (4)

Exercise 1. Find an example of a set X and a collection $S \subset 2^X$ that satisfies (5) and (6) but not (4).

We will often say simply “measurable” rather than ”μ-measurable”, when there is no hope of confusion.

A statement is true μ-a.e. if there exists a set $E \subset X$ such that $\mu(E) = 0$, and the statement is true in $X \setminus E$.

A measure μ on a set X is said to be *regular* if for every $A \subset X$, there exists a measurable $B \subset X$ such that $A \subset B$ and $\mu(A) = \mu(B)$.

In practice, every measure that we encounter will be regular.

Also, we note:

Lemma 1. *Given an arbitrary meausre μ on a space X, there exists a regular measure μ^* such that if B is any μ-measurable set, then B is μ^*-measurable, and $\mu(B) = \mu^*(B)$.*

Proof. If we define

$$\mu^*(B) := \inf\{\mu(A) \ : \ A \ \mu\text{-measurable}, B \subset A\},$$

then it is straightforward to check that μ^* has the stated properties. $\square$

Exercise 2. Fill in the missing details in the proof.

From the lemma we can see that the question of whether or not a measure is regular has to do with how it behaves on unmeasurable sets, and that a measure μ is regular if “$\mu(B)$ is as large as possible for every unmeasurable B”

Regularity is a convenient condition to assume, since for example it allows one to drop the hypothesis of measurability in part *iii* of Theorem 1. This might, for example, save one from the necessity of verifying measurability in some arguments. But if we are chiefly interested in measurable sets, then the lemma suggests that we need not worry much about regularity.

Here is another easy

Exercise 3. Given any set X containing at least 2 points, construct a measure on X that is not regular.

2. Borel measures

We now assume that X is a topological space. (Before long, we will limit our attention to the case when $X = \mathbb{R}^n$ with the standard topology.)

A measure μ on X is Borel if every open set is μ-measurable

The Borel σ-algebra is the smallest σ-algebra containing the open set. A set belonging to this σ-algebra is said to be a Borel set. Thus, we could equivalently define a Borel measure to be one for which every Borel subset is measurable.

A measure μ is Borel regular if it is Borel and, in addition. for every $A \subset X$, there exists a Borel $B \subset X$ such that $A \subset B$ and $\mu(A) = \mu(B)$.

(This is not quite the same thing as being both Borel and regular. For example, "Borel regular" is a very restrictive condition on a topological space X on which the only two open sets are $\emptyset$ and X.)

In practice, every measure that we encounter will be Borel regular.

All the remarks made above about regular measures apply with small changes to Borel regular measures. In particular, we have

Lemma 2. *Given a Borel measure μ on a topological space X, there exists a Borel regular measure μ^{**} such that $\mu(B) = \mu^{**}(B)$ for every Borel set B.*

Proof. Define
$$\mu^*(B) := \inf\{\mu(A) \ : \ A \text{ Borel}, B \subset A\},$$
and then argue as before. $\square$

2.1. Caratheodory's criterion.

The following theorem gives a beautiful and simple characterization of Borel measures on metric spaces.

Theorem 2. (Carathéodory's Criterion) *If X is a metric space and μ is a measure on X, then μ is a Borel measure if and only if*
$$\mu(A \cup B) = \mu(A) + \mu(B) \ \textit{whenever dist}(A, B) > 0. \tag{7}$$

Here $\text{dist}\,(A, B) := \inf_{a\in A, b\in B} d(a, b)$, where d is the metric on X.

Proof. The proof that every Borel measure satisfies (7) is easy. Given A, B such that $\text{dist}(A, B) = \alpha > 0$, let $C := \cup_{a\in A}\{x \in X : d(x, a) < \alpha/2\}$. Then C is open and hence measurable, and so
$$\mu(A \cup B) \ = \ \mu((A \cup B) \cap C) + \mu((A \cup B) \setminus C) \ = \ \mu(A) + \mu(B).$$

We now assume that (7) holds, and we give the proof of the other implication. It suffices to prove that every closed set is measurable. Let C be a closed set and A an arbitrary set. We must show that
$$\mu(A) \geq \mu(A \cap C) + \mu(A \setminus C).$$
We may assume that $\mu(A)$ is finite, as the above inequality is otherwise obvious.

For each j, let $C_j := \{x \in X \ : \ \text{dist}(x, C) \leq 1/j\}$. Then
$$\text{dist}\,(A \cap C, A \setminus C_j) > 0,$$
and so
$$\mu(A) \geq \mu((A \cap C) \cup (A \setminus C_j) = \mu(A \cap C) + \mu(A \setminus C_j).$$

So it suffices to show that

$$\mu(A \setminus C) \;\le\; \lim_{j\to\infty} \mu(A \setminus C_j). \tag{8}$$

To do this, let $R_j := C_j \setminus C_{j+1}$, so that for every j

$$A \setminus C \;= (A \setminus C_j) \cup (\cup_{k=j}^{\infty} A \cap R_k)$$

(this is where we use the assumption that C is closed). Thus the subadditivity of μ implies that

$$\mu(A \setminus C) \le \; \mu(A \setminus C_j) + \sum_{k=j}^{\infty} \mu(A \cap R_k)$$

by (1). Thus to prove (8) we only need to show that

$$\sum_{k=j}^{\infty} \mu(A \cap R_k) \le \lim_{N\to\infty} \sum_{k=1}^{N} \mu(A \cap R_k) < \infty. \tag{9}$$

This follows by noting that $\mathrm{dist}(R_j, R_k) > 0$ whenever $|j-k| \ge 2$. Thus for every N

$$\begin{aligned}
\sum_{k=1}^{N} \mu(A \cap R_k) &\le \sum_{k=1}^{N} \mu(A \cap R_{2k}) + \sum_{k=1}^{N} \mu(A \cap R_{2k-1}) && \text{by (1)}\\
&= \mu(\cup_{k=1}^{N}(A \cap R_{2k})) + \mu(\cup_{k=1}^{N}(A \cap R_{2k-1})) && \text{using (7)}\\
&\le 2\mu(A) < \infty.
\end{aligned}$$

□

Exercise 4. Taking X to be the unit interval, construct a measure μ on X that is not Borel.

2.2. **An important family of examples.** Hausdorff measure, which we define below, plays a central role in geometric measure theory

First, for $s \ge 0$, let

$$\omega_s := \frac{\pi^{s/2}}{2^s \Gamma(\frac{s}{2}+1)}.$$

If I have gotten the formula right, whenever k is an integer, $\omega_k d^k$ is the k-dimensional Lebesgue measure of a Euclidean ball in $\mathbb{R}^k$ of diameter d (ie, radius $d/2$.)

Let X be a separable metric space. For $A \subset X$ and $s, \delta > 0$ we define

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum \omega_s (\mathrm{diam}\ C_i)^s \;:\; A \subset \cup C_i, \mathrm{diam}\ C_i \le \delta \;\; \forall i \right\}. \tag{10}$$

Then s-dimensional Hausdorff measure $\mathcal{H}^s$ is defined by

$$\boxed{\mathcal{H}^s(A) := \lim_{\delta\to 0} \mathcal{H}^s_\delta(A) = \sup_{\delta>0} \mathcal{H}^s_\delta(A).} \tag{11}$$

We make a number of remarks:

Remark 2. The reason we assume that X is separable is to guarantee that, given an arbitrary set $A \subset X$, it is easy to find countable sequences of sets C_i such that $A \subset \cup C_i$. We will in fact chiefly be interested in the case $X = \mathbb{R}^n$.

Remark 3. It is easy to see that if $X = \mathbb{R}^n$ with the usual metric, then

$$\begin{aligned}\mathcal{H}^s_\delta(A) &= \inf\left\{\sum \omega_s(\text{diam } C_i)^s \ : \ A \subset \cup C_i, \text{diam } C_i \le \delta \ \ \forall i, C_i \text{ closed} \,\forall i\right\} \\ &= \inf\left\{\sum \omega_s(\text{diam } C_i)^s \ : \ A \subset \cup C_i, \text{diam } C_i \le \delta \ \ \forall i, C_i \text{ open} \,\forall i\right\} \\ &= \inf\left\{\sum \omega_s(\text{diam } C_i)^s \ : \ A \subset \cup C_i, \text{diam } C_i \le \delta \ \ \forall i, C_i \text{ convex} \,\forall i\right\}.\end{aligned}$$

These follow from the fact that for any set $C \subset \mathbb{R}^n$,

$$\text{diam}(C) = \text{diam}(\bar{C}) = \inf\{\text{diam}(U) : U \text{ open}, C \subset U\} = \text{diam}(\text{co } C),$$

where co C denotes the convex hull of C.

The first two equalities in fact remain valid on any metric space X.

Remark 4. Note that the definition of $\mathcal{H}^s$ makes sense even if X is an infinite-dimensional space. In fact it is perfectly reasonable to try to measure finite-dimensional subsets of, for example, function spaces such as $L^2(\mathbb{R}^n)$.

Remark 5. It is a fact that $\omega_0 = 1$, and taking this for granted, one sees that $\mathcal{H}^0$ is counting measure, which means that

$$\mathcal{H}^0(A) = \begin{cases} \text{number of points in } A, & \text{if finite} \\ +\infty & \text{if not .} \end{cases}$$

Remark 6. $\mathcal{H}^s_\delta$ is *not* in general a Borel measure.

Remark 7. If $X = \mathbb{R}^n$ for some n then

$$\mathcal{H}^s(x + A) = \mathcal{H}^s(A)$$

for all $x \in \mathbb{R}^n$ and $A \subset \mathbb{R}^n$, and

$$\mathcal{H}^s(\lambda A) = \lambda^s \mathcal{H}^s(A)$$

where $x + A = \{x + a \ : \ a \in A\}$ and $\lambda A \ = \ \{\lambda a : a \ inA\}$.

Exercise 5. verify some or all of the above remarks.

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Lemma 3. *$\mathcal{H}^s$ is a Borel regular measure.*

Proof. It is easy to check that $\mathcal{H}$ satisfies the hypotheses (7) of Carathéodory's criterion, Theorem 2. Hence $\mathcal{H}^s$ is a Borel measure.

To show that $\mathcal{H}^s$ is Borel regular, it we must show that given any $A \subset X$, there exists a Borel set $B \supset A$ such that $\mathcal{H}^s(B) \le \mathcal{H}^s(A)$. Given $A \subset X$, construct B as follows: first, for every $\delta > 0$ let $\{C_i^\delta\}$ be a collection of closed sets such that $A \subset \cup C_i^\delta$ and $\sum \omega_s(\text{diam}\, C_i^\delta)^s \le \mathcal{H}^s_\delta(A) + \delta$. The existence of such a sequence of closed sets follows from Remark 3. Then define

$$B = \cap_{k=1}^\infty \cup_{i=1}^\infty C_i^{\delta_k} \qquad \text{for some sequence } \delta_k \searrow 0.$$

It is immediate that B is Borel, and also easy to see that $\mathcal{H}^s(B) \le \mathcal{H}^s(A)$. □

Lemma 4. *Suppose that $0 < s < t$, and let A be a subset of a separable metric space X. Then*

if $\mathcal{H}^s(A) < \infty$, then $H^t(A) = 0$.
if $\mathcal{H}^t(A) > 0$, then $H^s(A) = +\infty$.

The two conclusions of the lemma are easily seen to be equivalent. The redundancy is for emphasis.

Proof. If $\{C_i\}$ is any sequence of sets with $\operatorname{diam} C_i < \delta$ for all i and $A \subset \cup C_i$, then

$$\sum \omega_t(\text{diam } C_i)^t \;=\; \sum \omega_t(\text{diam } C_i)^{t-s}(\text{diam } C_i)^s \le \frac{\omega_t}{\omega_s}\delta^{t-s}\sum \omega_s(\text{diam } C_i)^s.$$

It follows that $\mathcal{H}^t_\delta(A) \le \frac{\omega_t}{\omega_s}\delta^{t-s}\mathcal{H}^s_\delta(A)$ This easily implies the conclusions of the lemma. $\square$

In view of the previous lemma, the following definition makes sense:

Definition 1. The Hausdorff dimension of a set $A \subset X$ is

$$\dim_{\mathrm{H}}(A) \;:=\; \inf\{t \ge 0 : \mathcal{H}^t(A) = 0\}.$$

We will later prove that $\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$, where $\mathcal{L}^n$ denotes Lebesgue measure. It follows that $\mathcal{H}^s(A) = +\infty$ if $s < n$ and A is any open set, or more generally any set of positive $\mathcal{L}^n$ measure.

2.3. **approximation properties, Egoroff's Theorem, and Lusin's Theorem for Borel measures.** Next we prove some results that are a little technical but very useful. For example, they are needed to prove that Lusin's Theorem (maybe more obviously a useful result) holds for Borel regular measures on metric spaces.

We begin with a useful lemma

Lemma 5. *Assume that X is a metric space, and that $S \subset 2^X$ is a collection of sets containing all the closed sets, and such that if $A_1, A_2, \ldots \in S$, then*

$$\cup_{i=1}^\infty A_i \in S. \qquad \textit{and} \qquad \cap_{i=1}^\infty A_i \in S. \tag{12}$$

Then S contains the Borel sets.

(Compare Exercise 1.)

From the proof one can easily see that the lemma is still true if S is assumed to contain all open sets, rather than all closed sets.

Proof. Let S_1 be the smallest family of subsets of S that contains the closed sets and satisfies (12). Then define

$$S_2 := \{A \subset X : A \in S_1 \text{ and } X \setminus A \in S_1\}.$$

Then it is straightforward to check that S_2 is a σ-algebra. Moreover, we claim that S_2 contains all the closed sets. To prove this, we must check that every closed set belongs to S_1, which is immediate, and that every open set belongs to S_1. This is clear, since given an open set U, we can write

$$U = \cup_{k=1}^\infty \{x \in U : \text{dist}(x, \partial U) \ge \frac{1}{k}\}$$

and the sets on the right-hand side are all closed. $\square$

Our first result addresses the possibillity of approximating a set from the inside by a closed set.

Theorem 3. *Suppose that μ is a Borel measure on a metric space X. Then for any Borel set E such that $\mu(E) < \infty$,*

$$\mu(E) = \sup \{\mu(C) \ : \ C \subset E, \quad C \text{ closed}\}. \tag{13}$$

If $X = \mathbb{R}^n$ then in fact

$$\mu(E) = \sup \{\mu(K) \ : \ K \subset E, \quad K \text{ compact}\}.$$

Proof. Fix a Borel set E. We would like to prove (13). It is convenient to define

$$\nu = \mu \llcorner E, \qquad \text{so that } \nu(A) := \mu(A \cap E).$$

Then $\nu(X) < \infty$ (this is what we gain by considering ν instead of μ), and it suffices to show that

$$\nu(A) = \sup \{\nu(C) \ : \ C \subset A, \quad C \text{ closed}\}. \tag{14}$$

for every Borel set $A \subset X$ (since then in particular it holds for $A = E$.)

To do this, we will write S to denote the collection of all subsets of $A \subset X$ that satisfy (14). In view of Lemma 5, to prove (14) it suffices to show that

- S contains every closed set.
- S is closed with respect to countable union.
- S is closed with respect to countable intersection.

The first of these is clear.

For countable unions: fix $\varepsilon > 0$ and assume that $A_1, A_2, \ldots$ belong to S. For each i, by definition of S, we can find a closed set C_i such that $C_i \subset A_i$, and

$$\nu(A_i \setminus C_i) = \nu(A_i) - \nu(C_i) < \varepsilon 2^{-i}$$

Here and below we use the fact that if $C \subset A$ and C s measurable, then $\nu(A) = \nu(C) + \nu(A \setminus C)$, which follows directly from the definition of measurability. Then

$$\begin{aligned}
\lim_{N \to \infty} \nu(\cup_{i=1}^{\infty} A_i \setminus \cup_{i=1}^{N} C_i) &= \nu(\cup_{i=1}^{\infty} A_i \setminus \cup_{i=1}^{\infty} C_i) \\
&\le \sum_{i=1}^{\infty} \nu(A_i \setminus C_i) \\
&< \varepsilon.
\end{aligned}$$

Hence there exists some N such that

$$\nu(\cup_{i=1}^{\infty} A_i) - \nu(\cup_{i=1}^{N} C_i) = \nu(\cup_{i=1}^{\infty} A_i \setminus \cup_{i=1}^{N} C_i) < \varepsilon.$$

Since $\cup_{i=1}^{N} C_i$ is a closed subset of $\cup_{i=1}^{\infty} A_i$ and ε is arbitrary, we conclude that $\cup_i A_i$ belongs to S.

For countable intersections: this is similar to countable unions (but actually a little easier.)

Exercise 6. supply the details for countable intersections.

The exercise completes the proof of (13).

For $X = \mathbb{R}^n$, we write $B(R, x) := \{y \in \mathbb{R}^n : |y - x| < R\}$. Then if $\mu(E) < \infty$ and E is measurable,

$$\begin{aligned}\mu(E) &= \sup_{k\in\mathbb{N}} \mu(E \cap \bar{B}(k, 0)) \\ &= \sup_{k\in\mathbb{N}} \sup\{\mu(C) : C \text{ closed}, C \subset E \cap \bar{B}(k, 0)\} \\ &= \sup\{\mu(K) : K \text{ compact}, K \subset E\}.\end{aligned}$$

□

Next we prove a complementary result about approximation from the outside by open sets.

Lemma 6. *Suppose that μ is a Borel measure on a metric space X, and that E is a Borel set such that*

$$E \subset \cup_{i=1}^{\infty} V_i, \qquad \text{with } V_i \text{ open and } \mu(V_i) < \infty \text{ for all } i.. \tag{15}$$

Then

$$\mu(E) = \inf\{\mu(O) : \ O \text{ open}, E \subset O\}. \tag{16}$$

Clearly (16) cannot hold if for example $\mu(E) < \infty$ and $\mu(O) = +\infty$ for every open set O (such as is the case with s-dimensional Hausdorff measure on $\mathbb{R}^n$ when $s < n$.) This example illustrates the necessity for some sort of hypothesis along the lines of (15).

Proof. For each V_i, we apply the previous theorem to find a closed set $C_i \subset V_i \setminus E$ such that

$$\mu((V_i \setminus E) \setminus C_i) < \varepsilon 2^{-i}.$$

Then we define

$$W = \cup_i (V_i \setminus C_i)$$

Clearly W is open, and since $W \setminus E \subset \cup_i (V_i \setminus E)$, we deduce from the choice of C_i that $\mu(W \setminus E) < \varepsilon$. □

We next prove

Theorem 4 (Egoroff's Theorem)**.** *Let μ be a measure on a space X, and assume that $f_i, i = 1, 2, \dots$ and g are measurable functions on X such that $f_i \to g$ μ almost everywhere. Assume also that A is a μ-measurable subset of X, and that $\mu(A) < \infty$.*

Then for any $\varepsilon > 0$ there is a μ-measurable set $E \subset A$ such that

$$f_j \to g \text{ uniformly on } E, \qquad \mu(A \setminus E) < \varepsilon.$$

We emphasize that the theorem assumes *only* that X is a set and μ a measure — we do not require any additional conditions here. So this is a less deep result that Lusin's Theorem, which follows.

Proof. For $i, j \geq 1$, let

$$E_{i,j} := \{x \in A : |f_k(x) - g(x)| < \frac{1}{i} \text{ for all } k \geq j\}.$$

Then for every i,

$$\lim_{j\to\infty} \mu(A \setminus E_{i,j}) = 0$$

by assumption, and so we can find some $J(i)$ such that

$$\mu(A \setminus E_{i,J(i)}) < \varepsilon 2^{-i}.$$

We define

$$E = \cap_{i=1}^{\infty} E_{i,J(i)}.$$

Then the definition implies that $f_j \to g$ uniformly on E, since $|f_j - g| < \frac{1}{i}$ on E for $j \geq J(i)$. And

$$\begin{aligned} \mu(A \setminus E) = \mu(A \setminus \cap_{i=1}^{\infty} E_{i,J(i)}) &= \mu(\cup_{i=1}^{\infty}(A \setminus E_{i,J(i)})) \\ &\leq \sum_{i=1}^{\infty} \mu(A \setminus E_{i,J(i)}) < \varepsilon. \end{aligned}$$

□

Theorem 5 (Lusin's Theorem)**.** *Assume that X is a Borel regular measure on a metric space X, that $f : X \to \mathbb{R}^m$ is μ-measurable, and that A is a μ-measurable set with $\mu(A) < \infty$.*

Then for every $\varepsilon > 0$ there exists a closed set $C \subset A$ such that the restriction of f to C is continuous, and

$$\mu(A \setminus C) < \varepsilon.$$

Proof. First we note that the general case easily follows from the case $m = 1$, so we focus on that for simplicity.

We may also assume that A is a Borel set, as otherwise we can replace it by a Borel set $\tilde{A}$ such that $A \subset \tilde{A}$ and $\mu(A) = \mu(\tilde{A})$.

Step 1 We now verify f is the characteristic function of a Borel set E satisfies the conclusion of the theorem. (In fact, as the proof shows, this case of Lusin's Theorem is more or less equivalent to Theorem 3, about inner approximation of measurable sets by closed sets.) Indeed, fix $\varepsilon > 0$, and use Theorem 3 to find closed subsets C_0 and C_1 of $A \cap E$ and $A \setminus E$ respectively, such that $\mu((A \cap E) \setminus C_0) + \mu((A \setminus E) \setminus C_1) < \varepsilon$. Then let $C = C_0 \cup C_1$. Let us write f_C to denote the restriction of f to C. We claim that f_C is continuous. It suffices to check that the inverse image by f_C of every closed set is closed. Since f_C takes on only two values, 0 and 1, we need to check that $C_0 = (f_C)^{-1}\{0\}$ and $C_1 = (f_C)^{-1}\{1\}$ are both closed in C, and this is immediate.

Step 2. It is easy to deduce from Step 1 that a finite linear combination of characteristic functions (ie, a function with finite range) satisfies the conclusion of the theorem.

Step 3. One can also check that, given any measurable function f, one can find a sequence f_i such that $f_i \to f$ a.e. in A, and each f_i has finite range. (we gave a slightly clumsy argument for this in the lecture.)

For each such f_i, given $\varepsilon > 0$, we can use Step 2 to find a closed set C_i such that f_{i,C_i} is continuous and $\mu(A \setminus C_i) < \varepsilon 2^{-i-1}$. Then every f_i is continuous on $\cap C_i$, and $\mu(A \setminus \cap C_i) < \varepsilon/2$. We can further use Egoroff's Theorem to find a closed set $C \subset (\cap C_i)$ such that $f_i \to f$ uniformly on C, and

$$\mu(A \setminus C) = \mu(A \setminus (\cap C_i)) + \mu((\cap C_i) \setminus C) < \varepsilon$$

Then f_C is a uniform limit of continuous functions, and hence continuous. □

3. MEASURABLE FUNCTIONS

Note, above we have taken for granted basic facts about measurable functions, such as the definition:

If μ is a measure on a set X, then a function $f : X \to \mathbb{R}$ is said to be *μ-measurable* if $f^{-1}(A)$ is μ-measurable for every open $A \subset \mathbb{R}$.

Of course the same definition makes sense for functions $f : X \to Y$ where Y is any topological space.

We recall without proof the following basic result.

Theorem 6. *Assume that μ is a measure on a set X.*

- *A linear combination of μ-measurable functions is μ-measurable.*
- *A product of μ-measurable functions is μ-measurable.*
- *if $f_1, f_2, \dots$ are μ-measurable functions, then so are*

$$\sup_i f_i, \qquad \inf_i f_i, \qquad \limsup_{i\to\infty} f_i, \qquad \liminf_{i\to\infty} f_i$$

- *If particular, if (f_i) is a sequence of μ-measurable functions, and $f_i \to f$ μ a.e., then f is μ-measurable.*