Mat 1501 lecture notes, Sept 16-20, 2013

1. Radon measures

A measure μ is said to be *locally compact* if

$$\mu(K) < \infty \text{ if } K \text{ is compact.} \tag{1}$$

A *Radon measure* on $\mathbb{R}^n$ is a Borel regular, locally compact measure.

1.1. approximation and Lusin-type properties. The following facts can be deduced from our earlier discussion about Borel regular measures.

Many of them are immediate consequences of our earlier results if $\mu(\mathbb{R}^n) < \infty$. If not, then the proofs may require writing $\mathbb{R}^n$ as a countable union of pieces of finite μ-measure, applying the suitable earlier theorem on each piece, and assembling the results in some fashion. In this way one can prove for example

Lemma 1. *If μ is a Radon measure on $\mathbb{R}^n$, then for any measurable set A,*

$$\mu(A) = \sup\{\mu(K) : A \supset K \text{ compact}\}$$

and for every A, *measurable or not,*

$$\mu(A) = \inf\{\mu(O) : A \subset O \text{ open.}\}$$

Second, from Lusin's Theorem,

Lemma 2. *If μ is a Radon measure on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^m$ is a μ-measurable function for any $\varepsilon > 0$ there exists C closed such that the restriction of f to C is continuous and $\mu(\mathbb{R}^n \setminus C) < \varepsilon$.*

Another consequence of Lusin's Theorem is

Lemma 3. *Assume that μ is a Radon measure on $\mathbb{R}^n$, that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a μ-measurable function, and that A is a μ-measurable set such that $\mu(A) < \infty$.*

Then for any $\varepsilon > 0$, there exists a continuous $\bar{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\mu(\{x \in A : f(x) \neq \bar{f}(x)\}) < \varepsilon.$$

This can proved by using Lusin's Theorem to find a compact subset $K \subset A$ such that $\mu(A \setminus K) < \varepsilon$ and the restriction of f to K is continuous, and then using the following extension lemma:

Lemma 4. *Assume that K is a compact subset of $\mathbb{R}^n$ and that $f : K \to \mathbb{R}^m$ is continuous. Then there exists a continuous function $\bar{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that $\bar{f} = f$ in K.*

PROOF. It is rather easy to reduce to the case $m = 1$. For this case, recall that for any continuous function $f : K \to \mathbb{R}$ with K compact, there exists a continuous nondecreasing function $\omega : [0, \infty) \to [0, \infty)$ such that $\omega(0) = 0$ and

$$|f(x) - f(y)| \le \omega(|x - y|) \text{ for all } x, y \in K.$$

In fact, $\omega(s) := \max\{|f(x) - f(y)| \ : x, y \in K, \ |x - y| \le s\}$ can be seen to have all the stated properties. Then we define

$$\bar{f}(x) = \min_{y \in K} \Big(f(y) + \omega(|y - x|) \Big)$$

and check that $\bar{f}$ is continuous and agrees with f in K. $\square$

In fact Lemma 3 is also true on all of $\mathbb{R}^n$. The proof involves, as with Lemma 2, first arguing on compact sets and then piecing together the results somehow. This leads to

Lemma 5. *Assume that μ is a Radon measure on $\mathbb{R}^n$, that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a μ-measurable function.*

Then $\forall\ \varepsilon > 0$, there exists a continuous $\bar{f} : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\mu(\{x \in \mathbb{R}^n : f(x) \neq \bar{f}(x)\}) < \varepsilon.$$

Exercise 1. Prove Lemma 5, taking for granted Lemma 3.

Finally, one can also deduce from Lusin's Theorem the following:

Lemma 6. *Assume that μ is a Radon measure on $\mathbb{R}^n$ and that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a μ-integrable function.*

Then for every $\varepsilon > 0$ there exists a continuous $g : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\int |f - g|\ d\mu < \varepsilon.$$

Proof. Recall that, by the dominated convergence theorem, for any $\varepsilon > 0$, there exists $M > 0$ such that

$$\int_{A_M} |f| d\mu < \frac{\varepsilon}{4}, \qquad \text{for } A_M := \{x \in \mathbb{R}^n : |f(x)| > M\}.$$

Then by Lemma 5, one can find a continuous $g : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\mu(\{x \in \mathbb{R}^n \setminus A_M : f(x) \neq g(x)\}) \leq \mu(\{x \in \mathbb{R}^n : f(x) \neq g(x)\}) < \frac{\varepsilon}{4M}. \tag{2}$$

We may assume that $|g(x)| \leq M$ everywhere, since otherwise we can replace g by

$$\begin{cases} g(x) & \text{if } |g(x)| \leq M \\ M\frac{g(x)}{|g(x)|} & \text{if not,} \end{cases}$$

and it will still be continuous and satisfy (2). Then

$$|f - g| \leq |f| + |g| \leq 2M \text{ in } \mathbb{R}^n \setminus A_M$$

and

$$|f - g| \leq |f| + |g| \leq |f| + M \leq 2|f| \text{ in } A_M\ .$$

Using these facts, we find that

$$\begin{aligned} \int_{\mathbb{R}^n} |f - g| d\mu &= \int_{A_M} |f - g| d\mu + \int_{\mathbb{R}^n \setminus A_M} |f - g| d\mu \\ &\leq \frac{\varepsilon}{2} + 2M\ \mu(\{x \in \mathbb{R}^n \setminus A_M : f(x) \neq g(x)\}) < \varepsilon. \end{aligned}$$

$\square$

1.2. Representation theorems. We introduce the notation

$$C_c(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R} : \ f \text{ is continuous with compact support}\}.$$

We will abbreviate in various ways, eg writing $C_c(\mathbb{R}^n)$ or possibly even just C_c, if there is no hope of confusion.

We will also use the notation $C_c^+(\mathbb{R}^n) := \{f \in C_c(\mathbb{R}^n) : f(x) \geq 0 \ \text{ for all } x\}$.

Given a Radon measure μ, we can define a linear functional on $C_c(\mathbb{R}^n)$, given by

$$f \in C_c(\mathbb{R}^n) \mapsto \lambda(f) = \int_{\mathbb{R}^n} f d\mu \in \mathbb{R}.$$

Such functionals satisfy a natural *positivity* condition, stated in (3) below. We first prove that the converse is true: every positive linear functional on $C_c(\mathbb{R}^n)$ can be represented, in this way, by a Radon measure. (Hence, this is an example of a *representation theorem.*)

Theorem 1. *Suppose that $\lambda : C_c(\mathbb{R}^n) \to (-\infty, \infty)$ be a linear mapping which is positive in the sense that*

$$\lambda(f) \geq 0 \quad \textit{for } f \in C_c^+(\mathbb{R}^n). \tag{3}$$

Then there exists a Radon measure μ on $\mathbb{R}^n$ such that

$$\lambda(f) \ = \ \int_{\mathbb{R}^n} f \ d\mu \tag{4}$$

For $A \subset \mathbb{R}^n$ we will use the notation $\mathbf{1}_A$ for the characteristic function of A, defined by

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if not.} \end{cases}$$

PROOF. **Step 0**: First note that (3) and the linearity of λ imply that

$$\lambda(f) \geq \lambda(g) \quad \text{for } f, g \in C_c(\mathbb{R}^n) \text{ such that } f \geq g. \tag{5}$$

Step 1: We define a measure μ on $\mathbb{R}^n$ as follows: given any open set O, we define

$$\mu(O) := \sup\{\lambda(f) \ : f \in C_c(\mathbb{R}^n), \ 0 \leq f \leq 1, \operatorname{supp} f \subset O\}.$$

Next, given an arbitrary set A we define

$$\mu(A) := \inf\{\mu(O) \ : \ O \text{ open}, \ A \subset O\}.$$

One easily checks from the definitions and (5) that

$$\text{If } f \in C_c^+(\mathbb{R}^n) \text{ and } f \leq \mathbf{1}_A, \text{ then } \lambda(f) \leq \mu(A). \tag{6}$$

and similarly

$$\text{If } f \in C_c^+(\mathbb{R}^n) \text{ and } f \geq \mathbf{1}_A, \text{ then } \lambda(f) \geq \mu(A). \tag{7}$$

We now verify that μ is a Radon measure. We must verify

measure

Borel — follows from Caratheodory's criterion

regular — suffices to show that an arbitrary A is contained in a Borel set of equal measure. This follows easily from the construction of μ;

locally finite — This follows from (7), since given any compact set K we can find $f \in C_c^+(\mathbb{R}^n)$ such that $f \geq \mathbf{1}_K$, and then $\mu(K) \leq \lambda(f) < \infty$..

Step 2. Fix $f \in C_c^+(\mathbb{R}^n)$. To check that (4) holds, we introduce some notation. Temporarily fix $\varepsilon > 0$, and for $k = 0, 1, 2, \ldots$ define

$$S_k^\varepsilon := \{x \in \operatorname{supp} f : f(x) > \varepsilon k\}.$$

Note that these are all open sets. For $k \geq 0$ let χ_k^ε denote the characteristic function of S_k^ε.

And for $k \geq 0$, define

$$f_k^\varepsilon(x)) := \begin{cases} 0 & \text{if } f(x) \leq \varepsilon k \\ f(x) - \varepsilon k & \text{if } \varepsilon k < f(x) \leq \varepsilon(k+1) \\ \varepsilon & \text{if } f(x) > \varepsilon(k+1). \end{cases}$$

Then $f_k^\varepsilon \in C_c^+(\mathbb{R}^n)$ and

$$\varepsilon \chi_{k+1}^\varepsilon \leq f_k^\varepsilon \leq \varepsilon \chi_k^\varepsilon \qquad f = \sum_{n=0}^\infty f_k^\varepsilon.$$

It follows that

$$\varepsilon \sum_{k=1}^\infty \chi_k^\varepsilon \ \leq \sum_{k=0}^\infty f_k^\varepsilon = f \ \leq \varepsilon \sum_{k=0}^\infty \chi_k^\varepsilon$$

So using (6) and (7), we see that

$$\int (\varepsilon \sum_{k=1}^\infty \chi_k^\varepsilon) d\mu \ = \ \varepsilon \sum_{k=1}^\infty \mu(\chi_k^\varepsilon) \ \leq \lambda(f) \ \leq \varepsilon \sum_{n=0}^\infty \mu(\chi_k^\varepsilon) \leq \int (\varepsilon \sum_{k=0}^\infty \chi_k^\varepsilon) d\mu.$$

If we let ε tend to zero, then $\varepsilon \sum_{k=1}^\infty \chi_k^\varepsilon - f$ and $\varepsilon \sum_{k=0}^\infty \chi_k^\varepsilon - f$ tend to zero uniformly and vanish outside the support of f, so the above inequalities yield

$$\int_{\mathbb{R}^n} f\, d\mu \leq \lambda(f) \leq \int_{\mathbb{R}^n} f \ d\mu.$$

$\square$

The next representation theorem drops the hypothesis of positivity and instead assumes only that λ is locally bounded, which is a weaker condition:

Theorem 2. *Suppose that $\lambda : C_c(\mathbb{R}^n) \to (-\infty, \infty)$ be a linear mapping which is locally bounded in the sense that for every compact set K there exists a constant C_K such that*

$$\lambda(f) \leq C_K \max_{x \in K} |f(x)| \quad \textit{whenever} \quad \operatorname{supp} f \subset K. \tag{8}$$

Then there exists a signed Radon measure μ on $\mathbb{R}^n$ such that

$$\lambda(f) \ = \ \int_{\mathbb{R}^n} f \ d\mu \tag{9}$$

for all $f \in C_c(\mathbb{R}^n)$.

Remark 1. The space $C_c(\mathbb{R}^n)$ is often given a topology which is characterized by the property that $f_k \to f$ in $C_c(\mathbb{R}^n)$ if and only if

- there exists some compact $K \subset \mathbb{R}^n$ such that $\operatorname{supp}(f_k) \subset K$ for all k, *and*
- $f_k \to f$ uniformly as $k \to \infty$.

One can then check that continuous linear functionals on $C_c(\mathbb{R}^n)$ are precisely those linear functionals that satisfy (8). If we accept this as a fact, then we can see that the above theorem identifies the space of Radon measures on $\mathbb{R}^n$ as the topological dual space of $C_c(\mathbb{R}^n)$.

Note that $C_c(\mathbb{R}^n)$ is *not* a Banach space with this topology. Indeed, although the sup norm appears in our definition of the topology (since the sup norm describes uniform convergence), it is easy to see that $C_c(\mathbb{R}^n)$ is not complete with respect to sup norm convergence....

Exercise 2. A linear functional $\lambda : C_c(\mathbb{R}^n) \to \mathbb{R}$ is continuous if $\lambda(f_k) \to \lambda(f)$ whenever $f_k \to f$ in the $C_c(\mathbb{R}^n)$ topology described above.

Prove that a linear functional $\lambda : C_c(\mathbb{R}^n) \to \mathbb{R}$ is continuous if and only if it satisfies (8).

Note that "if" is almost immediate. One way to prove "only if" is to assume that (8) fails, and to prove that then λ is not continuous. And to prove failure of continuity, it certainly suffices to find a sequence f_k, all having support in some fixed compact set K, such that $f_k \to 0$ uniformly, but $|\lambda(f_k)|$ is bounded away from zero, as $k \to \infty$.

Proof. **Step 1**: Given λ as above, we define a new mapping $|\lambda| : C_c^+(\mathbb{R}^n) \to \mathbb{R}$ by stipulating that

$$|\lambda|(f) := \sup\{\lambda(g) \ : \ g \in C_c(\mathbb{R}^n), |g| \le f\}.$$

We can extend $|\lambda|$ to $C_c(\mathbb{R}^n)$ by writing $|\lambda|(f) = |\lambda|(f^+) - |\lambda|(f^-)$ for f of the form $f = f^+ - f^-$, with $f^\pm \in C_c^+(\mathbb{R}^n)$.

Hypothesis (8) implies that $|\lambda|(f)$ is finite for every $f \in C_c^+(\mathbb{R}^n)$.

We claim that $|\lambda|$ is linear. It is clear that $|\lambda|(cf) = c|\lambda|(f)$ for $c \in \mathbb{R}$. To verify that $|\lambda|(f_1 + f_2) = |\lambda|(f_1) + |\lambda|(f_2)$, it suffices to consider $f_1, f_2 \ge 0$. For two such functions we claim that

(10)
$$\{g \ : \ g \in C_c(\mathbb{R}^n), \ |g| \le f_1+f_2\} \ = \ \{g_1+g_2 : g_1, g_2 \in C_c(\mathbb{R}^n), \ |g_i| \le f_i \text{ for } i = 1, 2\}.$$

The inclusion $\supset$ is obvious. To prove the other inclusion, suppose that we are given $g \in C_c(\mathbb{R}^n)$ such that $|g| \le f_1 + f_2$. If we define

$$g_i = \frac{g f_i}{f_1 + f_2},$$

then $|g_i| \le f_i \frac{|g|}{f_1+f_2} = f_1$, and also clearly $g = g_1 + g_2$. This establishes (10).

Using (10), we check that for $f_1, f_2 \ge 0$,

$$\begin{aligned} |\lambda|(f_1 + f_2) = \ & \sup\{\lambda(g) : \ g \in C_c(\mathbb{R}^n), \ \ |g| \le f_1 + f_2\} \\ = \ & \sup\{\lambda(g_1 + g_2) : \ g_1, g_2 \in C_c(\mathbb{R}^n), \ |g_i| \le f_i \text{ for } i = 1, 2\} \\ = \ & \sup\{\lambda(g_1) + \lambda(g_2) : \ g_1, g_2 \in C_c(\mathbb{R}^n), \ |g_i| \le f_i \text{ for } i = 1, 2\} \\ = \ & |\lambda|(f_1) + |\lambda|(f_2). \end{aligned}$$

Step 2: Next define $\lambda^\pm(f) = \frac{1}{2}(|\lambda|(f) \pm \lambda(f))$, so that $\lambda = \lambda^+ - \lambda^-$.

We claim that $\lambda^\pm$ satisfy the hypotheses of Theorem 1. It is clear from Step 1 that $\lambda^\pm$ are linear, so we only need to prove that they are positive. To do this, note that for $f \in C_c^+(\mathbb{R}^n)$,

$$\lambda^+(f) = \sup\{\frac{1}{2}(\lambda(g) + \lambda(f)) \ : \ g \in C_c(\mathbb{R}^n), |g| \le f\}.$$

In particular, by making the choice $g = -f$ we deduce that

$$\lambda^+(f) \geq \frac{1}{2}[\lambda(-f) + \lambda(f)] = \frac{1}{2}\lambda(-f+f) = 0.$$

Similarly $\lambda^- \geq 0$. It follows from Theorem 1 that there exist measures μ^+, μ^- such that

$$\lambda^\pm(f) = \int f d\mu^\pm$$

for all $f \in C_c(\mathbb{R}^n)$. It follows that

$$\lambda(f) = \lambda^+(f) - \lambda^-(f) = \int f d\mu^+ - \int f d\mu^-$$

which proves (9) for $\mu = \mu^+ - \mu^-$. □

Remark 2. Another function space, very closely related to $C_c(\mathbb{R}^n)$, is the

$$C_0(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{R} \mid f \text{ is continuous}, f(x) \to 0 \text{ as } |x| \to \infty\}$$

on which we define the norm

$$\|f\|_{C_0} := \max_{x \in \mathbb{R}^n} |f(x)|.$$

This space is a Banach space – in particular, it is complete with respect to norm convergence. So its dual space, the space of bounded linear functionals $C_0(\mathbb{R}^n) \to \mathbb{R}$, is also a Banach space. It is not hard to deduce from Theorem 2 that the dual space of $C_0(\mathbb{R}^n)$ can be identified with the space of signed Radon measures on $\mathbb{R}^n$ with *finite total variation.* That is, the following holds:

Theorem 3. *Assume that λ is a linear functional on $C_0(\mathbb{R}^n)$ and that there exists some constant M such that such that*

$$\lambda(f) \leq M\|f\|_{C_0} \qquad \text{for all } f \in C_0(\mathbb{R}^n).$$

Then there exists a signed Radon measure $\mu = \mu^+ - \mu^-$ (where both μ^+ and μ^- are Radon measures) such that

$$\lambda(f) = \int_{\mathbb{R}^n} f \, d\mu, \qquad \text{and} \qquad \mu^+(\mathbb{R}^n) + \mu^-(\mathbb{R}^n) \leq M.$$

Conversely, every such signed Radon measure induces a bounded linear functional on $C_0(\mathbb{R}^n)$.

Exercise 3. Deduce Theorem 3 from Theorem 2 (and additional arguments as necessary).

1.3. vector-valued measures. Now we prove a substantial generalization of Theorem 2.

Throughout this section Y denotes a separable Banach space, with norm $\|\cdot\|_Y$.

We will write $C_c(\mathbb{R}^n; Y)$ to denote the space of continuous, compactly supported functions $\mathbb{R}^n \to Y$. Our goal is to identify the dual space of this space.

We will write Y^* to denote the space of bounded linear functionals $Y \to \mathbb{R}$, with the norm

$$\|\alpha\|_{Y^*} := \sup\{\langle \alpha, y \rangle \ : \ \|y\|_Y \leq 1\}.$$

Here α denotes a generic element of Y^*, and $\langle \alpha, y \rangle$ denotes the action of $\alpha \in Y^*$ on $y \in Y$.

Theorem 4. *Suppose that $\lambda : C_c(\mathbb{R}^n; Y) \to (-\infty, \infty)$ be a linear mapping which is locally bounded in the sense that for every compact set K there exists a constant C_K such that for $F \in C_c(\mathbb{R}^n; Y)$,*

$$\lambda(F) \le C_K \sup_{x \in K} \|F(x)\|_Y \qquad \text{whenever } \operatorname{supp} F \subset K. \tag{11}$$

Then there exists a Radon measure μ on $\mathbb{R}^n$ and a weak- μ-measurable[1] function $\sigma : \mathbb{R}^n \to Y^*$ such that for every $F \in C_c(\mathbb{R}^n; Y)$,*

$$\lambda(F) \;=\; \int_{\mathbb{R}^n} \langle \sigma(x), F(x) \rangle \, d\mu \tag{12}$$

and

$$\|\sigma(x)\|_{Y^*} = 1 \qquad \mu \text{ almost everywhere.} \tag{13}$$

Remark 3. A function $\sigma : \mathbb{R}^n \to Y^*$ is weak-* μ-measurable if $\sigma^{-1}(O)$ is μ-measurable whenever O is a weak-* open subset of Y^*. We recall that the weak-* topology on Y^* is the topology generated by the collection of sets of the form

$$\{\alpha \in Y^* \;:\; a < \langle \alpha, y \rangle \; < b\}, \qquad \text{for } \; a, b \in \mathbb{R}, \;\; y \in Y.$$

In particular, one can check that $\sigma : \mathbb{R}^n \to Y^*$ is a weak-* measurable if and only if

$$x \mapsto \langle \sigma(x), y \rangle \text{ is a measurable real-valued function for every } y \in Y,$$

and that this condition is enough to guarantee that the integral on the right-hand side of (12) is well-defined.

Remark 4. Given a σ and μ as in the statement of the theorem, we can associate to any measurable $A \subset \mathbb{R}^n$ an element of Y^*, which we can denote $\int_A \sigma d\mu$, and defined by

$$\left\langle (\int_A \sigma \, d\mu) \;,\; y \right\rangle := \int_A \langle \sigma(x), y \rangle d\mu.$$

One can therefore understand the theorem as stating that the dual of $C_c(\mathbb{R}^n; Y)$ is the space of "Y^*-valued Radon measures", that is, locally finite, countably additive maps $\{\text{Borel subsets of } \mathbb{R}^n\} \to Y^*$.... (i'm not sure whether the notion of "Borel regular" makes sense in this context; anyway, I don't see any reason to worry about it.....)

Remark 5. As with Theorems 2 and 3 above, this has a closely related version characterizing the dual space of $C_0(\mathbb{R}^n; Y)$, which denotes the space of continuous functions $F : \mathbb{R}^n \to Y$ with the property that $\|F(x)\|_Y \to 0$ as $|x| \to \infty$. As above, C_0 has the nice feature that it is a Banach space, with norm $\|F\|_{C_0} = \sup_{x \in \mathbb{R}^n} \|F(x)\|_Y$.

In fact, the dual of $C_0(\mathbb{R}^n; Y)$ consists exactly of those "Y^*-valued Radon measures" (that is, pairs μ, σ as described in the theorem) such that $|\mu|(\mathbb{R}^n) < \infty$.

Before giving the proof of Theorem 4, we consider certain special cases (arising from concrete choices of Y.)

[1] see Remark 3 below

Example 1. First suppose that $Y = \mathbb{R}^m$ for some m, with the Euclidean norm, which makes $Y = \mathbb{R}^m$ into a Hilbert space and allows us to identify Y and Y^*.

Then the theorem states that for every linear function $\lambda : C_c(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R}$ satisfying (12), there exists a Radon measure μ an a μ-measurable function $\sigma : \mathbb{R}^n \to \mathbb{R}^m$ such that $|\sigma| = 1$ μ a.e., and

$$\lambda(F) = \int_{\mathbb{R}^n} F \cdot \sigma d\mu.$$

We can also define $\mu_i = \sigma_i \mu$ for $i = 1, \ldots, m$ (a signed Radon measure in general), and then the above can be written

$$\lambda(F) = \sum_{i=1}^m \int_{\mathbb{R}^n} F_i \; d\mu_i = \int_{\mathbb{R}^n} F \cdot d\vec{\mu}$$

where $\vec{\mu} = (\mu_1, \ldots, \mu_m)$. It is clearly natural to think of $\vec{\mu}$ as a "vector-valued measure".

Note that when we speak of a vector-valued measure in this way, each component is actually a *signed* measure in general.

The case $m = n$ arises particularly often.

Example 2. A rather natural example involving an infinite-dimensional Banach space Y arises as follows.

We fix positive integers n, k, and we will write points in $\mathbb{R}^{n+k}$ in the form $(x, z) \in \mathbb{R}^n \times \mathbb{R}^k$.

For any function $F \in C_0(\mathbb{R}^{n+k})$ and any $x \in \mathbb{R}^n$, we will write F_x to denote the function $\mathbb{R}^k \to \mathbb{R}$ defined by

$$F_x(z) = F(x, z).$$

Note that $F_x \in C_0(\mathbb{R}^k)$ for every $x \in \mathbb{R}^n$, and moreover

$$\|F_{x_k} - F_x\|_{C_0(\mathbb{R}^k)} = \sup_{z \in \mathbb{R}^k} |F(x_k, z) - F(x, z)| \to 0 \text{ if } x_k \to x,$$

and

$$\|F_x\|_{C_0(\mathbb{R}^k)} = \sup_{z \in \mathbb{R}^k} |F(x, z)| \to 0 \text{ as } |x| \to \infty.$$

Thus in fact, the function $\mathbb{R}^n \to C_0(\mathbb{R}^k)$ described above is continuous and decays to zero at infinity; in other words, it belongs to $C_0(\mathbb{R}^n; C_0(\mathbb{R}^k))$.

So we have a map $C_0(\mathbb{R}^{n+k}) \to C_0(\mathbb{R}^n, C_0(\mathbb{R}^k))$, described above. A bit of thought, along the above lines, shows that this map is in fact a bijection. In addition, it is in fact a Banach space isomorphism, since

$$\begin{aligned}\|F\|_{C_0(\mathbb{R}^{n+k})} = \sup_{(x,z) \in \mathbb{R}^{n+k}} |F(x, z)| = \sup_{x \in \mathbb{R}^n} \sup_{z \in \mathbb{R}^k} |F(x, z)| &= \sup_{x \in \mathbb{R}^n} \|F_x\|_{C_0(\mathbb{R}^k)} \\ &= \|F\|_{C_0(\mathbb{R}^n; C_0(\mathbb{R}^k))}.\end{aligned}$$

In this way, we obtain a space of the form $C_0(\mathbb{R}^n, Y)$ by a simple change of perspective from the very standard space $C_0(\mathbb{R}^{n+k})$.

Now, suppose that ν is a *finite* signed Radon measure on $\mathbb{R}^{n+k}$, so that μ can be written $\nu = \nu^+ - \nu^-$, with $\nu^\pm(\mathbb{R}^{n+k}) < \infty$.

Then in view of Theorem 3, we can associate to ν a linear functional λ on $C_0(\mathbb{R}^{n+k})$.

But since $C_0(\mathbb{R}^{n+k})$ is isomorphic to $C_0(\mathbb{R}^n; \mathbb{C}_0(\mathbb{R}^k))$, we can view ν as a bounded linear functional on the latter space.

Then by applying Theorem 4 (or rather, the analogous theorem, where C_c is replaced by C_0) we conclude that there exists a finite Radon measure μ on $\mathbb{R}^n$ and a weakly measurable function $\sigma : \mathbb{R}^n \to C_0(\mathbb{R}^k)^*$ such that

$$\int_{\mathbb{R}^{n+k}} F \, d\nu \; = \; \lambda(F) \; = \; \int_{\mathbb{R}^n} \langle F_x, \sigma(x) \rangle d\mu(x)$$

and $\|\sigma\|_{C_0^*} = 1$ at μ a.e. $x \in \mathbb{R}^n$. Recalling that $C_0(\mathbb{R}^k)^*$ can be identified with the space of finite Radon measures on $\mathbb{R}^k$, it is reasonable to write the value of the function σ at the point $x \in \mathbb{R}^n$ as $\sigma_x(dz)$. With this notation, we conclude that

$$\int_{\mathbb{R}^{n+k}} F \, d\nu = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^k} F_x(z) \sigma_x(dz) \right) \mu(dx)$$

Thus, these rather abstract considerations yield a way of decomposing a measure ν in $\mathbb{R}^{n+k}$ that is sometimes useful.

Remark 6. Weak-* measurability is the natural notion of measurability in the setting of Theorem 4. Indeed, *strong* measurability of maps $\mathbb{R}^n \to Y^*$ (that is, measurability with respect to the norm topology on Y^*) often turns out be be *too* strong a condition, in the sense that many natural examples fail to be strongly measurable.

For example, consider functions from $\mathbb{R}$ into $C_0(\mathbb{R})^*$. Identifying the latter space with the space of Radon measures, an example of such a map is:

$$\sigma_x(dz) = \delta_{f(x)}(dz) \tag{14}$$

where $f : \mathbb{R} \to \mathbb{R}$ is any continuous function. Equivalently,

$$\langle \sigma_x, g \rangle = g(f(x)) \qquad \text{for every } g \in C_0(\mathbb{R}).$$

It is easy to check that functions $\sigma : \mathbb{R} \to C_0(\mathbb{R})^*$ of this form are weak-* measurable. However, even a very simple example of a function of the form (14) may fail to be strongly measurable. Indeed:

Exercise 4. Prove that the function $\sigma : \mathbb{R} \to C_0(\mathbb{R})^*$ defined by

$$\sigma_x(dz) = \delta_x(dz)$$

is not strongly measurable. In other words, show that there exists a subset of $C_0(\mathbb{R})^*$, open with respect to the norm topology, whose inverse image via σ is nonmeasurable.

Hint: The proof reduces to finding an unmeasurable $A \subset \mathbb{R}$ and a set $O \subset C_0(\mathbb{R})^*$, open with respect to the norm topology, such that

$$\{\delta_x(dz) \; : \; x \in A\} = O \cap \{\delta_x(dz) : x \in \mathbb{R}\}.$$

In fact, this can be done for *any* $A \subset \mathbb{R}$.

It is helpful to note that for any two distinct real numbers a and b,

$$\|\delta_a(dz) - \delta_b(dz)\|_{C_0(\mathbb{R})^*} = 2$$

Remark 7. The proof of Theorem 4 will use the fact that if μ is a Radon measure on $\mathbb{R}^n$ and λ is a linear functional on $C_c(\mathbb{R}^n)$ such that

$$|\lambda(f)| \le M \int |f| \, d\mu \qquad \text{for all } f \in C_c(\mathbb{R}^n),$$

then there exists a μ-measurable function g such that

$$|g(x)| \leq M \ \mu \text{ a.e., and} \qquad \lambda(f) = \int fg \ d\mu.$$

One way to prove this is to invoke our earlier results to find that there exists a signed measure ν such that $\lambda(f) = \int f d\nu$ for all f, and then argue (this is the part we will do later) that ν can be written $\mu \llcorner g$ for some g such that $\|g\|_{L^\infty(\mathbb{R}^n;\mu)} \leq 1$.

This fact is more or less a restatement of the fact that the dual space of $L^1(\mathbb{R}^n, d\mu)$ is $L^\infty(\mathbb{R}^n; d\mu)$, which you may have seen in some other context.

Proof of Theorem 4. **Step 1**. We start by defining, for $f \in C_c^+(\mathbb{R}^n)$ (that is, real-valued and nonnegative),

$$|\lambda|(f) := \sup\{\lambda(F) : F \in C_c(\mathbb{R}^n; Y), \ \|F(x)\|_Y \leq f(x) \text{ for all } x\}.$$

For general real-valued $f \in C_c(\mathbb{R}^n)$, we define $|\lambda|(f) = |\lambda|(f^+) - |\lambda|(f^-)$.

Then, by much the same arguments as in the proof of Theorem 2, one can verify that $|\lambda|$ is a positive linear functional on $C_c(\mathbb{R}^n)$. Thus by Theorem 1 there exists a Radon measure μ such that

$$|\lambda|(f) = \int_{\mathbb{R}^n} f d\mu \qquad \text{for all } f \in C_c(\mathbb{R}^n).$$

Step 2. Now let D denote a countable dense subset of Y, and assume that D is a vector space over $\mathbb{Q}$ (so that $ay_1 + by_2 \in D$ if $y_1, y_2 \in D$ and $a, b \in \mathbb{Q}$.

For each $y \in D$, we define a linear functional $\lambda_y : C_c(\mathbb{R}^n) \to \mathbb{R}$ by

$$\lambda_y(f) = \lambda(fy)$$

where fy dentotes the function $x \in \mathbb{R}^n \mapsto f(x)y \in Y$, which belongs to $C_c(\mathbb{R}^n; Y)$.

The definition of $|\lambda|$ and Step 1 imply that for every $f \in C_c(\mathbb{R}^n)$ and every $y \in D$,

$$|\lambda_y(f)| \leq \|y\|_Y \ |\lambda|(f) \ = \ \|y\|_Y \int f \ d\mu.$$

So by the fact stated in Remark 7, for every $y \in D$ there exists a μ-measurable function $g_y : \mathbb{R}^n \to \mathbb{R}$ such that $|g_y(x)| \leq M$ for μ a.e. x, and

$$\lambda_y(f) = \int_{\mathbb{R}^n} f \ g_y \ d\mu.$$

Step 3. We next claim that there exists a weakly μ-measurable function $\sigma : \mathbb{R}^n \to Y^*$ such that $\langle \sigma(x), y \rangle = g_y(x)$ for μ a.e. x.

To see this, note that the linearity of λ implies that

$$\lambda_{ay_1+by_2} = \lambda_{ay_2} + \lambda_{by_2} = a\lambda_{y_1} + b\lambda_{y_2} \tag{15}$$

μ a.e., for all $y_1, y_2 \in D$ and $a, b \in \mathbb{Q}$. (For every a, b, y_1, y_2, we may have to throw out a set of μ-measure zero on which this fails, but since there are only countable many combinations of a, b, y_1, y_2, we can find a set of full measure on which the identity (15) holds for *all* of them.) Thus

$$g_{ay_1+by_2}(x) = ag_{y_1}(x) + bg_{y_2}(x), \qquad |g_y(x)| \leq \|y\|_Y \tag{16}$$

μ a.e.. Fix some x at which this holds, an consider the map

$$y \in D \mapsto g_y(x).$$

This is a bounded linear real-valued function whose domain is a dense subset of Y. It therefore has a unique extension to all of Y, which we will write $\sigma(x) \in Y^*$, and which satisfies $\langle \sigma(x), y\rangle = g_y(x)$ for all $y \in D$ and $\|\sigma\|_{Y^*} \leq 1$.

In this way we define $\sigma(x)$ for μ a.e. $x \in \mathbb{R}^n$. The fact that σ is weakly μ-measurable can be inferred from the measurability of g_y, for all $y \in D$. Indeed, for any $y \in Y$, the function

$$x \mapsto \langle \sigma(x), y\rangle$$

is the limit (μ a.e.) of a sequence g_{y_k} for $(y_k) \subset D$ converging to y, and hence is measurable, as a pointwise limit of measurable functions.

Step 4. Next, given any $F \in C_c(\mathbb{R}^n; Y)$, we demonstrate that

$$\lambda(F) = \int_{\mathbb{R}^n} \langle \sigma(x), F(x)\rangle \, d\mu. \tag{17}$$

Note that for F of the form $F(x) = f(x)y$, where $y \in D$ and $f \in C_c(\mathbb{R}^n)$, we have already proved this identity in Steps 2 and 3 above.

To prove (17), observe that since F is continuous, and D is dense, for any $\varepsilon > 0$ we can find $y_i \in D$ and $f_i \in C_c(\mathbb{R}^n)$ for $i = 1, \dots L$ (with L depending on ε) such that

$$\sup_{x\in\mathbb{R}^n} \|F(x) - \sum_{i=1}^{L} f_i(x)y_i\|_Y < \varepsilon, \qquad \operatorname{supp}(\sum_{i=1}^{L} f_i(x)y_i) \subset \operatorname{supp}(F).$$

Then

$$\lambda(F) - \lambda(\sum_{i=1}^{L} f_i y_i) = \lambda(F - \sum_{i=1}^{L} f_i y_i) \leq \varepsilon K_{\text{supp F}}.$$

On the other hand,

$$|\int \langle \sigma, F\rangle d\mu - \lambda(\sum_{i=1}^{L} f_i y_i)| = |\int \langle \sigma, F - \sum_{i=1}^{L} f_i y_i\rangle d\mu| \leq \varepsilon \mu(\operatorname{supp}(F)).$$

By combining these and letting $\varepsilon \to 0$, we obtain (17).

Step 5. It remains to prove that $\|\sigma(x)\|_{Y^*} = 1$ at μ a.e.x.

We already know that $\|\sigma(x)\|_{Y^*} \leq 1$ a.e.. We thus only need to show that for every $\delta > 0$,

$$\mu(A_\delta) = 0 \qquad \text{for } A_\delta := \{x \in \mathbb{R}^n : \|\sigma(x)\|_{Y^*} \leq 1 - \delta\}.$$

We also remark that $x \mapsto \|\sigma(x)\|_{Y^*}$ is measurable, since

$$\|\sigma(x)\|_{Y^*} = \sup\{\langle \sigma(x), y\rangle \ : \ y \in D, \|y\| \leq 1\}.$$

Let O be an open set containing A_δ such that $\mu(O \setminus A_\delta) = \mu(O) - \mu(A_\delta) < \varepsilon$, for some $\varepsilon > 0$. Let

$$\mathcal{A} := \{F \in C_c(\mathbb{R}^n; Y) : \operatorname{supp}(F) \subset O, \|F(x)\|_Y \leq 1 \text{ for all } x\}.$$

Then

$$\begin{aligned}
\mu(A_\delta) \leq \mu(O) &= \sup\{\int f d\mu : C \in C_c(\mathbb{R}^n), f \leq \mathbf{1}_O\} \\
&= \sup\{|\lambda|(f) : C \in C_c(\mathbb{R}^n), f \leq \mathbf{1}_O\} \\
&\leq \sup_{F\in\mathcal{A}} \lambda(F).
\end{aligned}$$

However, for any $F \in \mathcal{A}$,

$$\begin{aligned}\lambda(F) &= \int_{O\setminus A_\delta} \langle \sigma(x), F(x)\rangle d\mu + \int_{A_\delta} \langle \sigma(x), F(x)\rangle d\mu \\ &\leq \mu(O \setminus A_\delta) \sup_{x\in\mathbb{R}^n} |\langle \sigma(x), F(x)\rangle| + \mu(A_\delta) \sup_{A_\delta} |\langle \sigma(x), F(x)\rangle| \\ &\leq \varepsilon + (1-\delta)\mu(A_\delta).\end{aligned}$$

Thus $\mu(A_\delta) \leq \varepsilon + (1-\delta)\mu(A_\delta)$. Since ε is arbitrary, this implies that $\mu(A_\delta) = 0$. $\square$