Mat 1501 lecture notes, week of September 23-27.

1. differentiation of measures and covering theorems

Recall that σ is absolutely continuous with respect to ν if $\sigma(A) = 0$ whenever $\nu(A) = 0$. When this holds we write $\sigma \ll \nu$. If μ is a signed measure, then $\mu \ll \nu$ means that μ can be written in the form $\mu^+ - \mu^-$, with $\mu^+, \mu^- \ll \nu$.

Recall also that measures σ and ν on a set Ω are mutually singular if there exists a set $B \subset \Omega$ such that $\sigma = \sigma \sqcup B$ and $\nu = \nu \sqcup (\Omega \setminus B)$. When this holds we write $\sigma \perp \nu$.

Thus we refer to μ_{ac} as the absolutely continuous part of μ , and μ_s as the singular part of μ (sometimes without explicitly mentioning the reference measure ν .)

Our next goal is to prove the following important result.

Theorem 1. Suppose that ν is a Radon measure, and μ a signed Radon measure, on \mathbb{R}^n . Then

- 1. $\lim_{r\to 0} \frac{\mu(B_r(x))}{\nu(B_r(x))} = \frac{d\mu}{d\nu}(x)$ exists and is finite for ν almost every x.
- 2. $\mu = \mu_{ac} + \mu_s$, where μ_{ac}, μ_s are signed Radon measures such that $\mu_{ac} \ll \nu$ and $\mu_s \perp \nu$. 3. $\mu_{ac}(B) = \int_B \frac{d\mu}{d\nu} d\nu$ for every Borel set B.

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Remark 1. One may be tempted to say that measures σ and ν are mutually singular iff their supports are disjoint. This is **not** correct, due to the way the support of a measure is defined. Indeed, in a topological space X

$$\operatorname{supp}(\sigma) = X \setminus \bigcup_{Vopen:\sigma(V)=0} V.$$

Thus in particular the support is always closed. In a metric space X, this is equivalent to

$$\operatorname{supp}(\sigma) = \{ x \in X : \sigma(B_r(x)) > 0 \text{ for all } r > 0 \}.$$

This for example, if $\{x_j\}$ and $\{y_j\}$ are two countable dense subsets of the unit interval $(0,1) \subset \mathbb{R}$ such that $\{x_j\} \cap \{y_j\} = \emptyset$, and if $\sigma = \sum 2^{-j} \delta_{x_j}$ and $\nu =$ $\sum 2^{-j} \delta_{y_i}$, then $\sigma \perp \nu$, but $\operatorname{supp}(\sigma) = \operatorname{supp}(\nu) = [0, 1]$. In general,

disjoint supports \Rightarrow mutually singular

(this should be clear) but as the above example shows, the converse is definitely not true.

If ν is a measure and f is a ν -integrable function, then $\nu {\,\sqsubseteq\,} f$ denotes the (in general signed) measure defined by

$$(\nu \, \llcorner \, f)(A) = \int_A f \, d\nu.$$

You are probably familiar with simpler results in the same spirit as Theorem 1, but with ν replaced by Lebesgue measure \mathcal{L}^n . Proofs of such results normally rely on the Vitali covering lemma. The utility of the Vitali covering lemma in these arguments stems from the fact that

$$\mathcal{L}^n(B_{5r}(x)) = 5^n \mathcal{L}^n(B_r(x))$$

for every $x \in \mathbb{R}^n$ and r > 0. For an arbitrary Radon measure μ , nothing like this is true, and so the Vitali covering lemma is not very useful. So our first task is to develop some argument that we can use in place of the Vitali covering lemma. That is the content of the following crucial lemma, whose proof we omit:

Lemma 1 (Besicovitch Covering Lemma). For every positive integer n, there exists a number P(n) with the following property:

Assume that A is a bounded subset of \mathbb{R}^n and that \mathcal{B} is a family of closed balls such that

(1) for every
$$x \in A$$
, $\inf\{r : B_r(x) \in \mathcal{B}\} = 0$.

Then there are families $\mathcal{B}^1, \ldots, \mathcal{B}^{P(n)}$ of closed balls (some of them possibly empty) such that, if we write $\mathcal{B}^i = \{B^i_i\}_j$, then

(2)
$$B_j^i \in \mathcal{B} \text{ for all } i, j$$

and

(4)
$$B_{i}^{i} \cap B_{i'}^{i} = \emptyset$$
 whenever $j \neq j'$.

The lemma allows us to cover a set A (the "set of centers") by at most P(n) families of *pairwise disjoint* closed balls. The point is that the amount of overlap is bounded by some absolute constant, depending only on the dimension. The proof can be found for example in the book of Mattila.

The Besicovitch covering lemma has the following consequence:

Lemma 2. Let μ be a finite Radon measure on \mathbb{R}^n , and suppose that $A \subset \mathbb{R}^n$, with $\mu(A) < \infty$. Let \mathcal{B} be a family of closed balls with the property that

(5) for every $x \in A$, $\inf\{r : B_r(x) \in \mathcal{B}\} = 0$.

Then there is a countable, pariwise disjoint collection of balls $\{B_i\} \subset \mathcal{B}$ such that

(6)
$$\mu(A \setminus \cup_i B_i) = 0.$$

Note that we do *not* need to assume that A is μ -measurable.

PROOF. We may assume that $\mu(A) > 0$, as otherwise the conclusion of the lemma is trivial.

Step 1: Fix an open set U such that $A \subset U$ and $\mu(U) \leq (1 + \varepsilon)\mu(A)$, for ε to be chosen. We discard from \mathcal{B} all balls that are not subsets of U; as remarked earlier, the modified collection of balls that we obtain in this way(we will abuse notation and still call it \mathcal{B}) continues to satisfy (5).

Let $\mathcal{B}^1, \ldots, \mathcal{B}^{P(n)}$ be collections of closed balls contained in \mathcal{B} satisfying the conclusions of the Besicovitch Covering Lemma. Then

$$\mu(A) \le \sum_{i=1}^{P(n)} \mu(\cup_j B_j^i)$$

by (3), so there exists some i_0 such that

$$\frac{1}{P(n)}\mu(A) \le \mu(\cup_j B_j^{i_0})$$

So by taking M_1 sufficiently large, we obtain

$$\frac{1}{2P(n)}\mu(A) \le \mu(\cup_{j=1}^{M_1} B_j^{i_0}).$$

Let us relabel these balls $\{B_1, \ldots, B_{M_1}\}$. Then

$$\mu(A \setminus \bigcup_{j=1}^{M_1} B_j) \le \mu(U \setminus \bigcup_{j=1}^{M_1} B_j) = \mu(U) - \mu(\bigcup_{j=1}^{M_1} B_j) \le \left[(1+\varepsilon) - \frac{1}{2P(n)} \right] \mu(A)$$

using the previous inequality and the choice of U at the last step.

Now fix $\varepsilon = \frac{1}{4P(n)}$ and $\theta = 1 - \frac{1}{4P(n)}$, so that the above inequality becomes

$$\mu(A \setminus \bigcup_{j=1}^{M_1} B_j) \le \theta \mu(A).$$

Step 2: Now we let $A_1 = A \setminus \bigcup_{j=1}^{M_1} B_j$, and we let U_1 be an open set containing A_1 and disjoint from $\bigcup_{j=1}^{M_1} B_j$, and such that $\mu(U_1) \leq \mu(A_1)(1+\varepsilon)$ for the value of ε chosen above. We repeat the above argument to find a new finite collection of closed, pairwise disjoint balls, say $\{B_{M_1+1}, \ldots, B_{M_2}\}$, that are contained in U_1 (and hence disjoint from $\{B_j\}_{j=1}^{M_1}$) and such that

$$\mu(A \setminus \cup_{j=1}^{M_2} B_j) = \mu(A_1 \setminus \cup_{j=M_1+1}^{M_2} B_j) \le \theta \mu(A_1) \le \theta^2 \mu(A).$$

Step 3. Repeating this procedure a countable number of times and taking the union of all balls found in the process, we obtain a pairwise disjoint collection of closed balls satisfying the required conclusion (6). \Box

Remark 2. When approximating a set A by closed disjoint balls, as in the above lemma, we generally want to arrange that the balls

- \bullet contain μ almost all of A this can be done under the hypotheses of Lemma 2
- and don't contain too much extra stuff.

To satisfy the latter condition, we can always fix an open set O containing A and such that $\mu(O \setminus A) < \varepsilon$. Then we can replace \mathcal{B} by

$$\mathcal{B}' := \{ B \in \mathcal{B} : B \subset O \}$$

It is not hard to see that \mathcal{B}' still satisfies (5), and any for any $\{B_i\} \subset \mathcal{B}'$, clearly $\mu(\cup_i B_i) \leq \mu(O) \leq \mu(A) + \varepsilon$, so that we can make the "extra stuff" covered by the balls arbitrarily small.

Of course this stratgy was used in the proof of the above lemma

We introduce some notation that will be used in the proof of the Theorem. For Radon measures μ and ν on \mathbb{R}^n and $x \in \mathbb{R}^n$ we define

(7)
$$\overline{D}(\mu,\nu,x) := \limsup_{r \to 0} \frac{\mu(B_r(x))}{\nu(B_r(x))}$$

(8)
$$\underline{D}(\mu,\nu,x) := \liminf_{r \to 0} \frac{\mu(B_r(x))}{\nu(B_r(x))}$$

Note that

Lemma 3. $\underline{D}(\mu, \nu, x), \overline{D}(\mu, \nu, x)$ are both Borel measurable functions.

PROOF. We recall that a real-valued function f is upper semicontinuous if $f^{-1}((-\infty, a))$ is an open set for every $a \in \mathbb{R}$. If the domain of f is a metric space for example, then this is equivalent to the condition that $f(x) \geq \limsup_{y \to x} f(y)$ for every $x \in X$. The main steps in the proof of the lemma are:

- (1) verify¹ that $x \mapsto \mu(B_r(x))$ is upper semicontinuous for every fixed r. The same of course holds for $x \mapsto \nu(B_r(x))$.
- (2) It is a standard fact from measure theory that upper semicontinuous functions are Borel measurable.
- (3) It is a standard fact from measure theory, that if f, g are nonnegative Borel measurable functions, then f/g is measurable. (Here we use the conventions $0/0 = 0, a/0 = \operatorname{sign}(a)\infty$.) Thus $x \mapsto \frac{\mu(B_r(x))}{\nu(B_r(x))}$ is a Borel function for every fixed r.
- (4) It is a standard fact from measure theory that if f_r is a measurable function for every r, then $\overline{f}(x) := \limsup_{r \to 0} f_r(x)$ and $\underline{f}(x) := \liminf_{r \to 0} f_r(x)$ are measurable functions.

Exercise 1. Carry out the proof of Step 1 in the above argument. Concerning Steps 2-4, refamiliarize yourself with the relevant standard facts from measure theory, as necessary.

Exercise 2. Construct a measure μ on a metric space X (for example a subset of some \mathbb{R}^n) containing a point x for which $\limsup_{y\to x} \mu(B_r(y)) < \mu(B_r(x))$.

Also, if you like, construct a measure μ on a metric space X for which there exists some $x \in X$ such that $\liminf_{y \to x} \mu(U_r(y)) > \mu(U_r(x))$. (see footnote for notation.)

The main ingredient in the proof of Theorem 1 is the following

Lemma 4. Assume that μ, ν are Radon measures on \mathbb{R}^n and that $0 \leq t < \infty$. Let $A \subset \mathbb{R}^n$. Then:

- 1. If $\overline{D}(\mu, \nu, x) \ge t$ for every $x \in A$ then $\mu(A) \ge t\nu(A)$.
- 2. If $\underline{D}(\mu, \nu, x) \leq t$ for every $x \in A$ then $\mu(A) \leq t\nu(A)$.

Here we do *not* require that A is either μ - or ν -measurable.

PROOF. We will prove conclusion (2), so we assume that A is a set such that $\underline{D}(\mu,\nu,x) \leq t$ for every $x \in A$. We may also assume that $\nu(A) < \infty$ as otherwise the conclusion is immediate.

Fix $\varepsilon > 0$ let U be an open set such that $A \subset U$ and $\nu(U) \leq \nu(A) + \varepsilon$.

Let $\mathcal{B} = \{B_r(x) : x \in A, B_r(x) \subset U, \frac{\mu(B_r(x))}{\nu(B_r(x))} \leq t + \varepsilon\}$. From the definitions it is clear that \mathcal{B} satisfies (5), so we conclude from Lemma 2 the existence of a countable

¹ The proof uses the fact that $B_r(x)$ is defined to be a *closed* ball. If we instead consider open balls, $U_r(x) := \{y : |x - y| < r\}$, then one can check that $x \mapsto \mu(U_r(x))$ is lower semicontinuous.

family $\{B_i\}$ of balls satisfying (6) for the measure μ . Using these balls we compute

$$\mu(A) = \mu(A \cap (\cup B_i)) \qquad \text{by (6)}$$

$$\leq \sum \mu(B_i) \qquad \text{by definition of } \mathcal{B}$$

$$\leq (t+\varepsilon) \sum \nu(U) \qquad \text{since } \{B_i\} \text{ are pairwise disjoint and } \cup B_i \subset U$$

$$\leq (t+\varepsilon)(\nu(A)+\varepsilon).$$

By letting ε tend to zero we deduce conclusion (2). The other conclusion is obtained in a similar way.

Exercise 3. Prove that

- 1. If $\overline{D}(\mu, \nu, x) \ge t$ for ν almost every $x \in A$ then $\mu(A) \ge t\nu(A)$.
- 2. If $\underline{D}(\mu, \nu, x) \leq t$ for μ almost every $x \in A$ then $\mu(A) \leq t\nu(A)$.

Exercise 4. Construct examples to show that it is *not* in general true that

- 1. If $\overline{D}(\mu, \nu, x) \ge t$ for μ almost every $x \in A$ then $\mu(A) \ge t\nu(A)$.
- 2. If $\underline{D}(\mu, \nu, x) \leq t$ for ν almost every $x \in A$ then $\mu(A) \leq t\nu(A)$.

We now give the

PROOF OF THEOREM 1. We will prove the theorem under the assumption that μ is a Radon measure.

Exercise 5. Prove the general case of the theorem (ie, in which μ is a *signed* Radon measure) follows, once we know the theorem holds when μ is a Radon measure.

Step 1. Define

$$\begin{split} &Z_0 := \{ x \in \mathbb{R}^n : \overline{D}(\mu,\nu,x) = +\infty \} \\ &Z_1 := \{ x \in \mathbb{R}^n : \underline{D}(\mu,\nu,x) < \overline{D}(\mu,\nu,x) \}. \end{split}$$

Step 1a. We first claim that

(9)

$$Z_0 := \bigcup_{R \in \mathbb{N}} \bigcap_{k \in \mathbb{N}} C_{R,k}, \qquad C_{R,k} := \{ x \in \mathbb{R}^n : |x| \le R, \overline{D}(\mu, \nu, x) \ge k \}$$

 $\nu(Z_0) = 0.$

And by Lemma 4,

$$\nu(C_{R,k}) \le \frac{1}{k}\mu(C_{R,k}) \le \frac{1}{k}\mu(\{x \in \mathbb{R}^n : |x| \le R\}.$$

Hence $\nu(\cap_{k\in\mathbb{N}}C_{R,k})=0$ for every R, which implies (9).

Step 1b. We next claim that

(10)
$$\nu(Z_1) = \mu(Z_1) = 0$$

Toward this end, for $0 \le s < t < \infty$ we define

$$A_{s,t,R} := \{ x \in \mathbb{R}^n : |x| \le R, \underline{D}(\mu,\nu,x) \le s < t \le \overline{D}(\mu,\nu,x) \}$$

Then Lemma 4 implies that

$$\mu(A_{s,t,R}) \le s\nu(A_{s,t,R}) \le \frac{s}{t}\mu(A_{s,t,R}).$$

$Z_1 = \bigcup_{R,s,t \text{ positive rationals }} A_{s,t,R}$

At this point we have completed the proof of conclusion (1) of Theorem 1, since it follows imediately from (9) and (10).

Step 2: Next, let $\mu_s := \mu \bigsqcup Z_0$, and let $\mu_{ac} = \mu \bigsqcup (\mathbb{R}^n \setminus Z_0)$. To prove conclusion (2) of the theorem, we must show that $\mu_s \perp \nu$ and $\mu_{ac} \ll \nu$. The first conclusion is clear from the definition of μ_s , in view of the fact that $\nu(Z_0) = 0$.

To prove that $\mu_{ac} \ll \nu$, note that if A is a Borel measurable set then

$$\mu_{ac}(A) = \bigcup_{m=1}^{\infty} \mu(\{x \in A : m-1 \le \overline{D}(\mu,\nu,x) < m\})$$
$$\leq \sum_{m=1}^{\infty} m \ \nu(\{x \in A : m-1 \le \overline{D}(\mu,\nu,x) < m\})$$

using Lemma 4 and the fact that $\underline{D}(\mu,\nu,x) \leq \overline{D}(\mu,\nu,x)$ at every x, which implies that $\underline{D}(\mu,\nu,x) \leq m$ everywhere in $\{x \in A : m-1 \leq \overline{D}(\mu,\nu,x) < m\}$.

In particular the above implies that $\mu_{ac}(A) = 0$ whenever $\nu(A) = 0$. Since μ is Borel regular, this same conclusion follows for also for non-measurable sets. Hence $\mu_{ac} \ll \nu$.

3. Finally we fix an arbitrary Borel set B, and we check that

(11)
$$\mu_{ac}(B) = \int_{B} \frac{d\mu}{d\nu} \, d\nu.$$

Let us write $Z = Z_0 \cup Z_1$. Then $\nu(Z) = 0$, so

$$\int_{B} \frac{d\mu}{d\nu} \, d\nu = \int_{B \setminus Z} \frac{d\mu}{d\nu} \, d\nu$$

Now fix t > 1, and for every integer k define

$$S_k := \{ x \in B \setminus Z : t^k \le \frac{d\mu}{d\nu} < t^{k+1} \},$$
$$S_{-\infty} := \{ x \in B \setminus Z : \frac{d\mu}{d\nu}(x) = 0 \}.$$

We will use the notation $\mathbb{Z}_* := \{-\infty\} \cup \mathbb{Z}$. Then $B \setminus Z = \bigsqcup_{k \in \mathbb{Z}_*} S_k$, since $\frac{d\mu}{d\nu}$ exists and is finite everywhere in $B \setminus Z$. It follows that

$$\int_{B} \frac{d\mu}{d\nu} \, d\nu = \int_{B \setminus Z} \frac{d\mu}{d\nu} \, d\nu = \int_{\bigsqcup_{k \in \mathbb{Z}_{*}} S_{k}} \frac{d\mu}{d\nu} \, d\nu = \sum_{k \in \mathbb{Z}_{*}} \int_{S_{k}} \frac{d\mu}{d\nu} \, d\nu.$$

By the definition of S_k ,

$$t^{k}\nu(S_{k}) \leq \int_{S_{k}} \frac{d\mu}{d\nu} d\nu \leq t^{k+1}\nu(S_{k}) \quad \text{if } k \in \mathbb{Z},$$
$$\int_{S_{-\infty}} \frac{d\mu}{d\nu} d\nu = 0.$$

However, since $\underline{D}(\mu,\nu,x) = \overline{D}(\mu,\nu,x) = \frac{d\mu}{d\nu}(x) \in [t^k,t^{k+1})$ in S_k , we can use Lemma 4 to find that

$$\mu(S_k) \leq t^{k+1}\nu(S_k), \qquad t^k\nu(S_k) \leq \mu(S_k).$$

Similarly, $\mu(S_{-\infty}) = 0$. Putting these together, we find that

$$\frac{1}{t}\sum_{k\in\mathbb{Z}_*}\mu(S_k)\leq \int_B\frac{d\mu}{d\nu}\ d\nu\leq t\sum_{k\in\mathbb{Z}_*}\mu(S_k).$$

Since

$$\sum_{k \in \mathbb{Z}_*} \mu(S_k) = \mu(\sqcup_{k \in \mathbb{Z}_*} S_k) = \mu(B \setminus Z),$$

we can let $t \searrow 1$ to conclude that

$$\int_B \frac{d\mu}{d\nu} \, d\nu = \mu(B \setminus Z).$$

Finally, we deduce (11) from this by noting that, since $\mu(Z_1) = 0$,

$$\mu(B \setminus Z) = \mu(B \setminus Z_0) = \mu \sqcup (\mathbb{R}^n \setminus Z_0)(B) = \mu_{ac}(B).$$

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2. applications of the above theorem

Now we record some consequences of the above result: **notation**: we write

$$\int_V f \ d\mu := \frac{1}{\mu(V)} \int_V f \ d\mu.$$

Corollary 1. If ν is a Radon measure on $U \subset \mathbb{R}^n$ and $f \in L^1_{loc}(U; d\nu)$, then

$$\lim_{r \to 0} \oint_{B_r(x)} f d\nu = f(x)$$

for ν almost every $x \in U$.

PROOF. We apply Theorem 1 to the signed measure $\mu = \nu \bigsqcup f$ and the Radon measure ν . Then $\frac{d\mu}{d\nu}(x)$ is exactly $\lim_{r\to 0} \int_{B_r(x)} f d\nu$. The result then follows, since $\nu \bigsqcup f = \mu = \nu \bigsqcup \frac{d\mu}{d\nu}$, so that $\frac{d\mu}{d\nu} = f$, ν a.e..

The above result can be strengthened as follows:

Corollary 2. If ν is a Radon measure on $U \subset \mathbb{R}^n$ and $f \in L^p_{loc}(U)$, then

$$\lim_{r \to 0} \oint_{B_r(x)} |f(x) - f(y)|^p \, d\nu(y) = 0$$

for ν almost every $x \in U$.

PROOF. Let $\{q_i\}$ be a countable dense subset of \mathbb{R} , and for each i, let E_i be a set of ν measure 0 such that

$$|f(x) - q_i|^p = \lim_{r \to 0} \oint_{B_r(x)} |f(y) - q_i|^p d\nu(y)$$

for every $x \in U \setminus E_i$; the existence of such a set follows from Corollary 1, since $x \mapsto |f(x) - a|^p$ is an integrable function for every $a \in \mathbb{R}$. Let $E = \bigcup_i E_i$. Then

 $\nu(E) = 0$, and if $x \notin E$ then

$$\begin{split} \limsup_{r \to 0} & \oint_{B_r(x)} |f(x) - f(y)|^p \, d\nu(y) \\ & \leq \limsup_{r \to 0} C \oint_{B_r(x)} |f(x) - q_i|^p + |q_i - f(y)|^p \, d\nu(y) \\ & \leq C |f(x) - q_i| \end{split}$$

for all *i*. The right-hand side can be made arbitrarily small, since $\{q_i\}$ is dense. \Box

Another corollary is a result already used in our discussion of representation theorems:

Corollary 3. Assume that ν is a Radon measure on \mathbb{R}^n and that $\lambda : C_c(\mathbb{R}^n) \to \mathbb{R}$ is a linear functional that satisfies

(12)
$$|\lambda(f)| \le M \int |f| \, d\nu$$
 for every $f \in C_c(\mathbb{R}^n)$.

Then there exists a ν -measurable function g such that

$$\lambda(f) = \int f g \, d\nu, \qquad |g(x)| \le M \text{ at } \nu \text{ a.e } x$$

This is a special case of a more general fact, but we only need the case of Radon measures on \mathbb{R}^n .

PROOF. (sketch)

First, it is easy to check that (12) implies that λ satisfies the hypotheses of one of our representation theorems, and hence that there exists a Radon measure μ such that

(13)
$$\lambda(f) = \int f d\mu \quad \text{for all } f \in C_c(\mathbb{R}^n).$$

We claim that

(14)

$$\mu \ll
u$$

(15)
$$\frac{d\mu}{d\nu}(x) \le M \text{ for } \nu \text{ a.e. } x.$$

Given (14) and (15), the conclusion of the corollary (with $g = \frac{d\mu}{d\nu}$) follows directly from Theorem 1.

Exercise 6. Prove (14) and (15).

Hint: Recalling from the proof of Theorem 1 that

$$\mu_{ac} = \mu \, \lfloor \, \{ x \in \mathbb{R}^n : \overline{D}(\mu, \nu, x) < \infty \},$$

both claims easily reduce to showing that $\overline{D}(\mu, \nu, x) \leq M$ at every x. To do this, it will be necessary to use the hypothesis (12) and the definition (13) of μ , since that is all we know about μ .

3.
$$\mathcal{H}^n = \mathcal{L}^n$$

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Our next result also involves the use of a covering lemma. We will prove

Theorem 2. $\mathcal{H}^n = \mathcal{L}^n$ on \mathbb{R}^n .

Recall that the definition of Hausdorff \mathcal{H}^s measure involves a constant ω_s chosen so that $\omega_k D^k$ is the volume (ie, k-dimensional Lebesgue measure) of a ball in \mathbb{R}^k of diameter D. (So ω_k is the lebesgue measure of a k-ball of radius 1/2.)

We will require an auxiliary result that we will state without proof.

Theorem 3 (Isodiametric Inequality). If C is a bounded subset of \mathbb{R}^n , then

$$\mathcal{L}^n(C) \le \omega_n (\operatorname{diam} C)^n.$$

Thus, of all sets with given diameter D, a ball encloses the largest volume.

The example of an equilateral triangle shows that a set of diameter D need not be contained in a ball of diameter D.

ABOUT THE PROOF. The theorem is proved by symmetrization. That is, the idea is to modify a given set C (of diameter D) in a series of steps that each preserve its volume (ie Lebesgue measure) and do not increase its diameter. In this way one can produce a set, say C_{sym} , that is symmetric with respect to reflection through all the coordinate hyperplanes, say of diameter $D_{sym} \leq D$. This set can be contained in a ball of diameter D, so

$$\mathcal{L}^{n}(C) = \mathcal{L}^{n}(C_{sym}) \le \omega_{n}(D_{sym})^{n} \le \omega_{n}D^{n}.$$

PROOF OF THEOREM 2. Step 1. First, fix $\delta > 0$ and $A \subset \mathbb{R}^n$, and note that by the isodiametric inequality,

$$\mathcal{H}^{n}_{\delta}(A) = \inf\{\sum \omega_{n}(\operatorname{diam} C_{i})^{n} : A \subset \cup C_{i}, \operatorname{diam} C_{i} < \delta\}$$

$$\geq \inf\{\sum \mathcal{L}^{n}(C_{i}) : A \subset \cup C_{i}, \operatorname{diam} C_{i} < \delta\}$$

$$\geq \mathcal{L}^{n}(A).$$

Letting $\delta \searrow 0$, we find that $\mathcal{H}^n \ge \mathcal{L}^n$.

Step 2. Recall that by definition,

$$\mathcal{L}^{n}(A) = \inf\{\sum |R_{i}| : A \subset \cup R_{i}, R_{i} \text{ a rectangle for all } i \}$$

where |R| denotes the product of the side-lengths of a rectangle, and rectangle here means one with sides parallel to the coordinate axes. (similarly for "cube" below.) It follows that

$$\mathcal{L}^{n}(A) = \inf\{\sum |Q_i| : A \subset \cup Q_i, Q_i \text{ a cube for all } i, \operatorname{diam} Q_i < \delta \text{ for all } i\}$$

Exercise 7. Check that this holds, if it is not obvious.

Taking this for granted, note that for every n, there is a (large) constant c_n such that $\omega_n(\operatorname{diam} Q)^n \leq c_n |Q|$ for every cube $Q \subset \mathbb{R}^n$. Thus for every $\delta > 0$,

$$\mathcal{L}^{n}(A) = \inf\{\sum |Q_{i}| : A \subset \cup Q_{i}, Q_{i} \text{ a cube for all } i, \operatorname{diam} Q_{i} < \delta \text{ for all } i\}$$

$$\geq \inf\{\sum \omega_{n} \frac{1}{c_{n}} (\operatorname{diam} Q)^{n} : A \subset \cup Q_{i}, Q_{i} \text{ as above}\}$$

$$\geq \frac{1}{c_{n}} \mathcal{H}^{n}_{\delta}(A)$$

for every A. Letting $\delta \searrow 0$, we conclude that

(16)
$$\mathcal{H}^n \le c_n \mathcal{L}^n$$

which is at least a step in the right direction.

Step 3. We next show that for any rectangle R,

(17)
$$|R| = \mathcal{L}^n(R) \ge \mathcal{H}^n(R).$$

To do this, fix $\delta > 0$, and note that by the Vitali covering theorem (for example), we can find a countable collection of pairwise disjoint closed balls $\{B_i\}_{i=1}^{\infty}$ such that

diam
$$B_i < \delta \ \forall i, \qquad \cup_i B_i \subset R, \qquad \mathcal{L}^n(R \setminus \cup_i B_i) = 0.$$

Also, in view of (16), $\mathcal{H}^n(R \setminus \bigcup_i B_i) = 0$, so for any $\varepsilon > 0$, there exist sets $C_i, i = 1, 2, \ldots$ such that

diam
$$C_i < \delta \ \forall i, \qquad R \setminus \bigcup B_i \subset \bigcup_i C_i, \qquad \sum \omega_n (\operatorname{diam} C_i)^n < \varepsilon.$$

Now $\{B_i\} \cup \{C_i\}$ constitute a covering of R by sets of diameter less than δ , so

$$\mathcal{H}^{n}_{\delta}(R) \leq \sum \omega_{n} (\operatorname{diam} B_{i})^{n} + \sum \omega_{n} (\operatorname{diam} C_{i})^{n}$$
$$\leq \sum \mathcal{L}^{n}(B_{i}) + \varepsilon.$$

Since the balls are pairwise disjoint and contained in R, it follow that

$$\mathcal{H}^n_\delta(R) \le \mathcal{L}^n(R) + \varepsilon$$

Letting $\delta \to 0$, we prove our claim (17).

Step 4. For an arbitrary $A \subset \mathbb{R}^n$, it follows that

$$\mathcal{L}^{n}(A) = \inf\{\sum_{i} |R_{i}| : A \subset \cup R_{i}, R_{i} \text{ a rectangle for all } i\}$$

$$\geq \inf\{\sum_{i} \mathcal{H}^{n}(R_{i}) : A \subset \cup R_{i}, R_{i} \text{ a rectangle for all } i\}$$

$$\geq \mathcal{H}^{n}(A).$$

4. weak convergence of measures

Let $\{\mu_n\}$ be a sequence of signed measures. We say that μ_n converges weakly as measures to a limit μ if

$$\lim_{n \to \infty} \int f d\mu_n = \int f d\mu$$

for all $f \in C_0(U)$. When this holds we write $\mu_n \rightharpoonup \mu$ weakly as measures, or (when no ambiguity can result) simply $\mu_n \rightharpoonup \mu$.

We began to discuss this topic. Notes to appear later.