**Mat 1501 lecture notes** October 28 - November 1

## 1. last remarks about $k$-vectors and covectors

A *unit simple $k$-vector* is a simple $k$-vector $v$ such that $|v| = 1$.

There is a one-to-one correspondence between unit simple $k$-vectors and oriented $k$-planes in $\mathbb{R}^N$.

To understand this, we first have to remember what we mean by an oriented $k$-plane.

Suppose that $P$ is any $k$-plane in $\mathbb{R}^N$ (or in any vector space) and that $\{v_i\}_{i=1}^k$ and $\{w_i\}_{i=1}^k$ are any two *ordered* bases for $P$.

Then there exists a $k \times k$ matrix $(a^i_j)$ such that $w_i = \sum_j a^j_i v_j$.

The two bases are said to have the same orientation if $\det(a^j_i) > 0$ and the opposite orientation otherwise.

Motivated by this, we say that an *oriented $k$-plane* is an equivalence class of $k$-tuples of linearly indepdent vectors, where

$$\{v_i\}_{i=1}^k \ \sim\ \{w_i\}_{i=1}^k \quad \text{if there exists a } k \times k \text{ matrix } a^j_i \text{ such that}$$
$$w_i = \textstyle\sum_j a^j_i v_j \text{ for all } i, \text{ and } \det(a^j_i) > 0.$$

The correspondence between unit simple $k$-vectors and oriented $k$-planes is summarized in the following fact:

$$(1) \quad \{v_i\}_{i=1}^k \ \text{ and } \ \{w_i\}_{i=1}^k \ \text{ represent the same oriented } k\text{-plane}$$
$$\Longleftrightarrow \qquad \frac{v_1 \wedge \ldots \wedge v_k}{|v_1 \wedge \ldots \wedge v_k|} = \frac{w_1 \wedge \ldots \wedge w_k}{|w_1 \wedge \ldots \wedge w_k|}$$

In fact, by using the alternating property of the exterior product, one can verify that if $w_i = \sum_j a^j_i v_j$ for all $i$, then

$$w_1 \wedge \ldots \wedge w_k = \det(a^j_i)\ v_1 \wedge \ldots \wedge v_k. \tag{2}$$

which makes the implication $\Rightarrow$ in (1) above straightforward. To prove $\Leftarrow$, one must first check that if $\{v_i\}_{i=1}^k$ and $\{w_i\}_{i=1}^k$ are $k$-tuples of linearly independent vectors and

$$w_1 \wedge \ldots \wedge w_k = \ c\ v_1 \wedge \ldots \wedge v_k \tag{3}$$

for some $c \in \mathbb{R}$ (necessarily nonzero) then $\{v_i\}$ and $\{w_i\}$ span the same $k$-plane, or equivalently, there exists some invertible $k \times k$ matrix $(a^j_i)$ such that $w_i = \sum_j a^j_i v_j$ for all $i$. To prove this, we can extend $\{v_1, \ldots, v_k\}$ to a basis $\{v_1, \ldots, v_N\}$ for $\mathbb{R}^N$. If we write $w_1, \ldots, w_k$ in terms of this basis, then (3) implies that in fact only $v_1, \ldots, v_k$ will appear in the expression for each $w_i$, which shows that indeed we can write $w_i = \sum_{j=1}^k a^j_i v_j$ for all $i$. Since $\text{span}\{w_i\}_{i=1}^k$ is $k$-dimensional by assumption, it is clear that $(a^i_j)$ is invertible. Then it is easy to complete the proof of the implication $\Leftarrow$ in (1) by again appealing to (2).

It is often convenient to choose an orthonormal basis for an oriented $k$-plane. Such bases will often be denoted $\tau_1, \ldots, \tau_k$, and in terms of such a basis, the corresponding $k$-vector is of course $\tau_1 \wedge \ldots \wedge \tau_k$.

## 2. differential forms

For $k \geq 1$, a $k$-form $\omega$ on $\mathbb{R}^N$ is a function defined on $\mathbb{R}^N$ such that $\omega(x) \in \Lambda^k(T_x\mathbb{R}^N)$ for every $x$.

We treat the case $k = 0$ differently: a 0-form is just a real-valued function on $\mathbb{R}^N$. (Or for a definition that is more consistent with the $k \geq 1$ definition, we can insist on the convention that $\Lambda_0 V$ and $\Lambda^0 V$ are 1-dimensional vector spaces for every $V$. We can then view a 0-form as a function $f$ such that $f(x) \in \Lambda_0(T_x\mathbb{R}^N)$ for every $x$, and we can identify this with a real-valued function.)

We will often, but not always[1], identify $T_x\mathbb{R}^N$ (for arbitrary $x \in \mathbb{R}^N$) with $\mathbb{R}^N$, in the usual way, and then we can view a $k$-form as a function from $\mathbb{R}^N$ into $\Lambda^k\mathbb{R}^N$.

**Example 1.** If $f$ is a smooth function = 0-form, then $df$ is the 1-form defined by

$$\langle df(x), v\rangle = \lim_{h\to 0} \tfrac{1}{h}[f(x+hv) - f(x)], \qquad \text{for } v \in T_x\mathbb{R}^N \cong \mathbb{R}^N.$$

In particular, for every $i = 1, \ldots, N$, we will write $dx^i$ to denote the 1-forms defined by

$$\langle dx^i, v\rangle = v_i \text{ for } v = (v_1, \ldots, v_N) \in \mathbb{R}^N.$$

Here we are abusing notation in a standard way by

- writing $x^i$ as an of abbreviation for the function $\pi^i : \mathbb{R}^N \to \mathbb{R}$ defined by $\pi^i(x_1, \ldots, x_N) = x_i$, and
- not explicitly recording the dependence of $dx^i$ on $x$, which however is kind of trivial (once we identify $T_x\mathbb{R}^N$ with $\mathbb{R}^N$ for all $x$.)

  Thus $\{dx^1, \ldots, dx^N\}$ (evaluated at some point $x$) is the basis for $\Lambda^1\mathbb{R}^N$ dual to the standard basis $\{e_1, \ldots, e_N\}$ (at the same point $x$), that ie, the analog of the standard dual basus $\omega^i$ from above.

If we introduce the notation

$$dx^\alpha = dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k} \qquad \text{for } \alpha \in I(N,k)$$

(where the exterior product in $\Lambda^*T_x\mathbb{R}^N \cong \Lambda^*\mathbb{R}^N$ is defined exactly as above) then it follows that every $k$-form can be written

$$\omega = \sum_{\alpha \in I(N,k)} a_\alpha dx^\alpha$$

for certain functions $a_\alpha : \mathbb{R}^N \to \mathbb{R}$.

In particular, we have

$$df = \sum_{i=1}^N \frac{\partial f}{\partial x_i} dx^i.$$

We say that a $k$-form is $C^\infty$ ($C^1, C^2$, etc) if $a_\alpha$ is $C^\infty$ ($C^1, C^2$ etc) for every $\alpha$, in the above representation. When we say "smooth", it generally will mean $C^\infty$.

If $U$ is an open subset of $\mathbb{R}^N$ then we will use the notation

$$\mathcal{E}^k(U) := \text{ the set of smooth } k\text{-forms on } U\ .$$

It is clear that $\mathcal{E}^k(U)$ is a vector space.

[1] we may allow ourselves to switch back and forth at will between two viewpoints, either identifying $T_x\mathbb{R}^N$ with $\mathbb{R}^N$ or not, as the situation requires. I hope that this will not cause any confusion.

**2.1. some notational conventions.**

- co-vectors such as the basis vectors $\omega^\alpha$ will always have superscripts, and $k$-vectors such as $e_\alpha$ will always have subscripts.
- as a result, $k$-forms (ie, functions taking values in spaces of covectors) such as $dx^\alpha$ will always have superscripts.
- One can also define $k$-vector-fields to be functions taking values in the space $\Lambda_k \mathbb{T}^N$, or (equivalently, up to the identification of $T_x\mathbb{R}^N$ with $\mathbb{R}^N$ for $x \in \mathbb{R}^N$) functions whose value at $x$ is an element of $\Lambda_k(T_x\mathbb{R}^N)$. I don't know if we will write these down much, but if we do, they will have subscripts.
- if we write a $k$-covector $\eta$ in terms of the basis $\{\omega^\alpha\}_{\alpha\in I(N,k)}$, then the component functions will always have subscripts. That is, we will always write
$$\eta = \sum_{\alpha\in I(N,k)} a_\alpha \omega^\alpha$$
with subscripts on the numbers $a_\alpha$.

  This is rather natural, if one thinks about it correctly, because as noted previously, we can find the coefficients $a_\alpha$ by letting $\omega$ act on a suitable $k$-vector:
$$a_\alpha = \langle \eta, e_\alpha\rangle.$$
So one can view $a_\alpha$ as inheriting the subscript $\alpha$ from $e_\alpha$.
- similarly, we write a $k$-vector $v \in \Lambda_k\mathbb{R}^N$ in the form
$$v = \sum_{\alpha\in I(N,k)} a^\alpha e_\alpha$$
with superscripts on the coefficients.
- And similarly for $k$-forms and $k$-vector-fields. For example, a $k$-form is always written
$$\omega = \sum_{\alpha\in I(N,k)} a_\alpha dx^\alpha.$$
where the coefficient functions have subscripts.

These conventions are the reason that, even if a point $x \in \mathbb{R}^N$ is written $(x_1, \ldots, x_N)$, we write $dx^i$ instead of $dx_i$.

**2.2. pullback.** Suppose that $f : \mathbb{R}^N \to \mathbb{R}^M$ is a smooth map.

Given a vector $v \in T_x\mathbb{R}^N$, we will write $f_*v$ to denote the vector in $T_{f(x)}\mathbb{R}^M$ defined by
$$f_*v = \nabla f(x)(v).$$

Similarly, for a multivector $v = v_1 \wedge \ldots v_k \in \Lambda_k T_x\mathbb{R}^N$, we will write $f_*v$ to denote the element of $\Lambda_k Tf(x)\mathbb{R}^M$ defined by
$$f_*v = f_*v_1 \wedge \ldots f_*v_k.$$
Next, given a $k$-form $\omega$ on $\mathbb{R}^M$, we write $f^*\omega$ to denote the $k$-form on $\mathbb{R}^N$ defined by
$$\langle f^*\omega(x), v\rangle = \langle \omega(f(x)), f_*v\rangle \qquad \text{for } v \in \Lambda_k T_x\mathbb{R}^N, \tag{4}$$
We call $f^*\omega$ the *pullback of* $\omega$ *by* $f$.

It is straightforward to check that if $y_1, \ldots, y_M$ denote coordinates on $\mathbb{R}^M$, and if

$$\omega(y) = \sum_{\alpha \in I(M,k)} a_\alpha(y) \; dy^\alpha$$

and if $f = (f^1, \ldots, f^M)$, then

$$f^*\omega(x) = \sum_{\alpha \in I(M,k)} a_\alpha(f(x)) \, df^{\alpha_1} \wedge \ldots \wedge df^{\alpha_k}$$

This can be rewritten very explicitly as shown below, although the simpler expression given above is often preferable.

$$\begin{aligned} f^*\omega(x) &= \sum_{\alpha \in I(M,k)} a_\alpha(f(x)) (\sum_{i_1=1}^{N} \frac{\partial f^{\alpha_1}}{\partial x_{i_1}} dx^{i_1}) \wedge \cdots \wedge (\sum_{i_k=1}^{N} \frac{\partial f^{\alpha_k}}{\partial x_{i_k}} \wedge dx^{i_k}) \\ &= \sum_{\alpha \in I(M,k)} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_\alpha(f(x)) \frac{\partial f^{\alpha_1}}{\partial x_{i_1}} \cdots \frac{\partial f^{\alpha_k}}{\partial x_{i_k}} dx^{i_1} \wedge \ldots \wedge dx^{i_k} \\ &= \sum_{\alpha \in I(M,k)} \sum_{\beta \in I(M,k)} a_\alpha(f(x)) \; \det(\frac{\partial f^{\alpha_i}}{\partial x_{\beta_j}}(x)) \; dx^\beta. \end{aligned}$$

One can also check that

$$f^*(a\omega + b\eta) = af^*\omega + bf^*\eta$$

and

$$f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta.$$

**2.3. exterior derivative.** Assume that $\omega = \sum_{\alpha \in I(N,k)} a_\alpha dx^\alpha$ is a smooth $k$-form on $\mathbb{R}^N$.

Then we define the *exterior derivative* $d\omega$ of $\omega$ to be the $(k+1)$-form

$$d\omega = \sum_{\alpha \in I(N,k)} \sum_{i=1}^{N} \frac{\partial a_\alpha}{\partial x_i} dx^i \wedge dx^\alpha$$

Note that this definition is consistent with the notation *df* for a 1-form introduced above, when $f$ is a 0-form.

Some basic properties of the exterior derivative are:

$$d(d\omega) = 0 \text{ for all } k\text{-forms } \omega, \qquad \text{or more briefly,} \qquad d^2 = 0.$$

$$d(a\omega + b\eta) = ad\omega + bd\eta.$$

$$\text{if } \omega \in \mathcal{E}^k(U) \text{ and } \eta \in \mathcal{E}^\ell(U), \text{ then } d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

Finally, we remark that *exterior differentiation commutes with pullback*:

$$df^*\omega = f^*d\omega \text{ for every } \omega \in \mathcal{E}^k(U).$$

All the above assertions are straightforward to check.

We remark that although our definition if the exterior derivative apparently depends on our choice of coordinates, in fact one can give a coordinate-free definition, and from this it is clear that the operation of exterior differentiation is independent of the choice of coordinates.

**2.4. integral of a $k$-form over a $k$-dmensional submanifold.** An $k$-dimensional *oriented submanifold* of $\mathbb{R}^N$ is a submanifold $M$ together with a map $\tau : M \to \Lambda_k \mathbb{R}^N$ such that $\tau(x)$ is a unit simple vector orienting $T_x M$, for every $x \in R$.

If $\omega$ is a $k$-form, then we define

$$\int_M \omega := \int_M \langle \omega(x), \tau(x) \rangle d\mathcal{H}^k(x). \tag{5}$$

This agrees with the standard differential geometry definition of $\int_M \omega$, which we recall goes as follows: First, using partitions of unity, we see that it suffices to define $\int_M \omega$ when $\omega$ is supported in a single coordinate chart. Thus, we may assume that $\mathrm{supp}(\omega) \cap M \subset f(V)$, where $V$ is an open subset of $\mathbb{R}^k$ and $f : V \to M \subset \mathbb{R}^N$ is a map (at least $C^1$) such that $\nabla f(x)$ has rank $k$ for all $x$, and moreover $\{f_* e_1, \ldots, f_* e_k\}$ is correctly oriented, so that

$$\frac{f_* e_1 \wedge \ldots \wedge f_* e_k}{|f_* e_1 \wedge \ldots \wedge f_* e_k|}(x) = \tau(f(x))$$

Recalling Lemma 7 from the previous set of notes, which characterizes the norm of a simple $k$-vector as a Jacobian, this in fact can be rewritten

$$f_* e_1 \wedge \ldots \wedge f_* e_k(x) = \tau(f(x)) \ Jf(x). \tag{6}$$

For such $\omega$ and $f$ the classical differential geometry definition is

$$\int_M \omega = \int_{f(V)} \omega =: \int_V f^* \omega.$$

For the right-hand side, recall from the definition of the integral of a $k$-form over an open subset of $\mathbb{R}^k$ that

$$\int_V f^* \omega \ := \ \int_V \langle f^* \omega, e_1 \wedge \ldots \wedge e_k \rangle d\mathcal{H}^k.$$

Thus

$$\begin{aligned}
\int_V f^* \omega = & \int_V \langle f^* \omega(x), e_1 \wedge \ldots \wedge e_k \rangle d\mathcal{H}^k \\
\overset{(4)}{=} & \int_V \langle \omega(f(x)), f_* e_1 \wedge \ldots \wedge f_* e_k \rangle d\mathcal{H}^k \\
\overset{(6)}{=} & \int_V \langle \omega(f(x)), \tau(f(x)) \rangle \ Jf(x) \ d\mathcal{H}^k \\
= & \int_{f(V)} \langle \omega(y), \tau(y) \rangle \ d\mathcal{H}^k(y)
\end{aligned}$$

by the area formula. Thus the two definitions are consistent. We will take (5) as our basic definition, however, since it is easier to extend to integraton over less smooth sets.

## 3. currents: basics

**3.1. definition of a current.** If $U$ is an open subset of $\mathbb{R}^N$, we define

$$\mathcal{D}^k(U) := \{\omega \in \mathcal{E}^k(U) : \ \omega \text{ has compact support in } U\}.$$

We define a topology on $\mathcal{D}^k(U)$ by specifying that $\omega^\ell \to \omega$ if

there exists a compact $K \subset U$ such that $\mathrm{supp}(\omega^\ell) \subset K$ for all $\ell$, and

$$(\frac{\partial}{\partial x_1})^{j_1}\cdots(\frac{\partial}{\partial x_N})^{j_N}\omega^\ell_\alpha \to (\frac{\partial}{\partial x_1})^{j_1}\cdots(\frac{\partial}{\partial x_N})^{j_N}\omega_\alpha$$

as $\ell \to \infty$, uniformly for every $\alpha \in I(N,k)$ and all nonnegative integers $j_1, \ldots, j_N$.

**Definition 1.** A $k$-dimensional current on $U$ is a continuous linear functional on $\mathcal{D}^k(U)$

The set of all such currents is denoted $\mathcal{D}_k(U)$.

If $M$ is an oriented $k$-dimensional submanifold of $\mathbb{R}^N$, then there is an associated $k$-current $[\![M]\!] \in \mathcal{D}_k(\mathbb{R}^N)$, defined by

$$[\![M]\!](\omega) = \int_M \omega.$$

All the definitions that follow are designed to reduce to classical definitions when considering currents of the form $[\![M]\!]$ above.

**3.2. the boundary of a current.** We next define the *boundary* of a current $T \in \mathcal{D}_k(U)$ to be the current $\partial T \in \mathcal{D}_{k-1}(U)$ defined by

$$\partial T(\omega) = T(d\omega) \qquad\qquad \text{for } \omega \in \mathcal{D}^{k-1}(U)$$

if $k \geq 1$. The standard convention is that $\partial T = 0$ for all $T \in \mathcal{D}_0(U)$.

This is motivated by Stokes' Theorem, which (together with the above definitions) implies that if $M$ is a oriented $C^1$ manifold-with-boundary, then

$$\partial[\![M]\!] = [\![\partial M]\!]. \tag{7}$$

It is clear that

$$\partial^2 = 0, \qquad \text{or more explicitly, } \partial(\partial T) = 0 \text{ for all } T \in \mathcal{D}_k(U). \tag{8}$$

**Exercise 1.** To familiarize yourself with the definitions, let $[\![(0,1)]\!]$ denote the 1-current associated to the unit interval $(0,1) \subset \mathbb{R}$ (with the standard orientation), and for every $p \in \mathbb{R}$ let $[\![p]\!]$ denote the 0-current associated to the 0-dimensional manifold $\{p\}$ (again, with the standard orientation.)

Verify, by writing out explicitly what everything means, that

$$\partial([\![(0,1)]\!]) = [\![1]\!] - [\![0]\!].$$

This is of course a special case of (7).

**3.3. the mass of a current.** Next, we define the *mass* of a current by

$$\mathbf{M}(T) := \sup\{T(\omega) \ : \ \omega \in \mathcal{D}^k(U), \|\omega\| \leq 1\}$$

where

$$\|\omega\| := \sup_{x \in U} |\omega(x)| := \sup_{x \in U} (\omega(x), \omega(x))^{1/2}.$$

More generally, if $W$ is an open subset of $U$, then we define

$$\mathbf{M}_W(T) := \sup\{T(\omega) \ : \ \omega \in \mathcal{D}^k(U), \|\omega\| \leq 1, \text{supp}(\omega) \subset W\}.$$

This definition has the property that if $M$ is a smooth submanifold of $U$ then

$$\mathbf{M}_W([\![M]\!]) = \mathcal{H}^k(M \cap W), \tag{9}$$

**Remark 1.** One should be aware that a slightly different definition of mass is often used. Let us temporarily call this $\tilde{\mathbf{M}}$. Then $\tilde{\mathbf{M}}$ is, roughly speaking, the *largest possible* notion of mass for which (9) still holds; this is sometimes a useful property.

$\tilde{\mathbf{M}}$ and $\mathbf{M}$ are equivalent in the sense that there exists some constant $C$ such that $\tilde{\mathbf{M}}(T) \le \mathbf{M}(T) \le C\tilde{\mathbf{M}}(T)$ for all $T \in \mathcal{D}_k(U)$.

Due to (9), $\tilde{\mathbf{M}}([\![M]\!]) = \mathbf{M}([\![M]\!])$ whenever $M$ is smooth. In fact the difference between $\tilde{\mathbf{M}}$ and $\mathbf{M}$ only becomes apparent when one considers currents that fail to be rectifiable, in language we will introduce below.

A current is $T$ said to have *locally finite mass* if $\mathbf{M}_W(T) < \infty$ for every open $W$ such that $\bar{W}$ is a compact subset of $U$.

**Proposition 1.** *Assume that $T$ is a $k$-current with locally finite mass in an open set $U \subset \mathbb{R}^N$. Then there is a Radon measure $\mu_T$ on $U$ and a $\mu_T$-measurable function $\vec{T} : U \to \Lambda_k(\mathbb{R}^N)$ such that*

$$T(\omega) = \int_U \langle \omega(x), \vec{T}(x) \rangle \, d\mu_T(x) \qquad \text{for all } \omega \in \mathcal{D}^k(U). \tag{10}$$

*In addition,*

$$|\vec{T}(x)| = 1, \qquad \mu_T \text{ almost everywhere}$$

*and*

$$\mathbf{M}_W(T) = \mu_T(W) \qquad \text{whenever } W \text{ is open.}$$

Proof. This is a direct consequence of our theorem about representation of linear functionals, once we note that any current $T$ with locally finite mass determines a bounded linear functional on $C_c(U; \Lambda^k \mathbb{R}^N)$. This is not hard to see, because

- At the outset, $T$ is a linear functional defined on $\mathcal{D}^k(U)$, which is a dense subset of $C_c(U; \Lambda^k \mathbb{R}^N)$,
- The hypothesis of finite mass says exactly that $T$, as defined on $\mathcal{D}^k(U)$, is continuous with respect to the topology of $C_c(U; \Lambda^k \mathbb{R}^N)$. Indeed, assume that $\omega^\ell$ is a sequence in $\mathcal{D}^k(U)$ such that $\omega^\ell \to \omega$ in the $C_c(U; \Lambda^k \mathbb{R}^N)$ topology. This means that there is a compact set $K$ such that $\text{supp}(\omega^\ell) \subset K$ for all $\ell$, and in addition $\|\omega^\ell - \omega\| \to 0$. Then by the definition of mass,

$$|T(\omega^\ell) - T(\omega)| = |T(\omega^\ell - \omega)| \le \|\omega^\ell - \omega\| \mathbf{M}_K(T) \quad \to \quad 0 \qquad \text{as } \ell \to \infty.$$

- Since $T$ is a linear functional defined on a dense subset of $C_c(U; \Lambda^k \mathbb{R}^N)$ and is continuous with respect to the $C_c(U; \Lambda^k \mathbb{R}^N)$ topology, it has a unique extension to a continuous linear functional $C_c(U; \Lambda^k \mathbb{R}^N) \to \mathbb{R}$, which we still denote $T$.

Applying the representation theorem to the linear functional $T : C_c(U; \Lambda^k \mathbb{R}^N) \to \mathbb{R}$ directly yields the conclusions of the proposition. $\square$

It follows from the Proposition, and the fact that 0-forms are just functions, that a 0-current with locally finite mass can be identified with a signed Radon measure.

Note also that if $\mu$ is any Radon measure on $U \subset \mathbb{R}^N$ and $\vec{T} : U \to \Lambda_k \mathbb{R}^N$ is any $\mu$-measurable function, then the map $T : \mathcal{D}^k(U) \to \mathbb{R}$ *defined* by (10) is a $k$-current.

A current is said to be *representable by integration* if it admits a representation of the form (10). The above discussion may be summarized by saying that a current is represntable by integration if and only if it has locally finite mass.

**Example 2.** Assume that

- $M \subset \mathbb{R}^N$ is a locally $k$-rectifiable set,
- $\theta : M \to \mathbb{N}$ is a locally $\mathcal{H}^n \llcorner M$ integrable function
- $\tau : M \to \Lambda_k \mathbb{R}^N$ is a measurable function such that for a.e. $x$, $\tau(x)$ orients $T_x M$, by which we mean that $\tau(x)$ can be written in the form $\tau_1 \wedge \ldots \wedge \tau_k$, where $\{\tau_i\}_{i=1}^k$ is an orthonormal basis for the approximate tangent space $T_x M$.

Then there is an associated current, defined by

$$T(\omega) := \int_M \langle \omega, \tau \rangle \theta \; d\mathcal{H}^k.$$

Currents of this form are called *integer multiplicity rectifiable currents.*

**Example 3.** As a special case of the above, note that a locally 0-rectifiable set is just a locally finite set of points in $\mathbb{R}^N$, say $\{p_i\}$. Locally finite means that $\#\{i : |p_i| < R\} < \infty$ for every $R > 0$.

Thus a integer multiplicity rectifiable 0-current in $\mathbb{R}^N$ is one of the form

$$T(f) = \sum_i \tau_i \theta_i f(p_i) \qquad f \in \mathcal{D}^0(\mathbb{R}^N) \cong C_c^\infty(\mathbb{R}^N),$$

where $\tau_i \in \{\pm 1\}$ and $\theta_i \in \mathbb{N}$. Equivalently, such a current can be written in the form

$$T = \sum_i a_i [\![ p_i ]\!] \qquad \text{for } a_i := \tau_i \theta_i \in \mathbb{Z}.$$

**Example 4.** Assume now that

- $M \subset \mathbb{R}^N$ is a countably $k$-rectifiable set,
- $\theta : \mathbb{R}^N \to [0, \infty)$ is locally $\mathcal{H}^n$-integrable and $M = \{x \in \mathbb{R}^N : \theta(x) > 0\}$,
- $\tau : M \to \Lambda_k \mathbb{R}^N$ is a measurable function such that for a.e. $x$, $\tau(x)$ orients $T_x M$ (the approximate tangent plane at $x$, with multiplicity $\theta(x)$, of the measure $\mathcal{H}^n \llcorner \theta$).

Then there is an associated current, defined by

$$T(\omega) := \int_M \langle \omega, \tau \rangle \theta \; d\mathcal{H}^k.$$

Currents of this form are called *rectifiable currents.*

**3.4. support and push-forward of a current.** The *support* of a current $T \in \mathcal{D}_k(U)$ is defined as

$$\begin{aligned} \operatorname{supp}(T) := \{ & x \in U : \text{for every neighborhood } O \text{ of } x, \\ & \text{there exists } \ \omega \in \mathcal{D}^k(U) \text{ such that } \operatorname{supp}(\omega) \subset O \text{ and } T(\omega) \neq 0. \}. \end{aligned}$$

Thus, $T(\omega) = 0$ for any $k$-form $\omega$ that vanishes in $\operatorname{supp}(T)$.

Assume that $T \in \mathcal{D}^k(U)$ and that $f : U \to V$ is a smooth map, for $U \subset \mathbb{R}^N$ and $V \subset \mathbb{R}^M$ open.

Assume also that $f$ is *proper*, by which we mean that $f^{-1}(K)$ is a compact subset of $U$ whenever $K$ is a compact subset of $V$. Then we define $f_*T \in \mathcal{D}^k(U)$ by

$$f_*T(\omega) = T(f^*\omega) \qquad \text{for all } \omega \in \mathcal{D}_k(V).$$

In fact, one can extend the definition to the case when the restriction of $f$ to $\mathrm{supp}(T)$ is proper, or equivalently, $f^{-1}(K) \cap \mathrm{supp}(T)$ is compact for every compact $K \subset V$. (For example, a situation that arises rather often is when $T$ has compact support and $f$ is a projection of $\mathbb{R}^N$ onto a lower-dimensional space, which is not a proper map.) In this situation we define

$$f_*T(\omega) = T(\chi f^*\omega) \qquad \text{for all } \omega \in \mathcal{D}_k(V) \text{ with } \mathrm{supp}(\omega) \subset K. \tag{11}$$

where $\chi$ is a compactly supported function such that $\chi = 1$ in a neighborhood of $f^{-1}(K) \cap \mathrm{supp}(T)$ .

**Exercise 2.** Verify that (11) is independent of the choice of $\chi$, so that $f_*T$ is in fact well-defined.

**Warning!** We will sometimes write "$T(f^*\omega)$" as an abbreviation for "$T(\chi f^*\omega)$, where $\chi$ is a compactly supported function etc etc...."

It is easy to verify that

$$\partial f_*T = f_*\partial T. \tag{12}$$

This is dual to the fact that exterior differentiation commutes with pullback.

The definition of push-forward is arranged so that if $M$ is a smooth submanifold and $f$ is a diffeomorphism, say, then

$$f_*[\![M]\!] = [\![f(M)]\!].$$

**3.5. weak convergence of currents.** A sequence $(T_\ell) \subset \mathcal{D}_k(U)$ is said to *converge weakly* if

$$T_\ell(\omega) \to T(\omega) \qquad \text{for all } \omega \in \mathcal{D}^k(U).$$

when this holds we write

$$T_\ell \rightharpoonup T \qquad \text{as } \ell \to \infty.$$

**Exercise 3.** Prove that if $(T_\ell)$ is a sequence of currents such that

$$\sup_\ell \mathbf{M}_W(T_\ell) < \infty \text{ for every bounded open } W$$

and if $T_\ell \rightharpoonup T$ as $\ell \to \infty$, then $T$ has locally finite mass in $U$, and

$$\mathbf{M}_W(T) \le \liminf_{\ell\to\infty} \mathbf{M}_W(T_\ell). \tag{13}$$

**Exercise 4.** Give an example of a sequence $(T_\ell)$ of $k$-currents with locally finite mass in $U \subset \mathbb{R}^N$ (you get to choose $k$ and $N$), and a limiting current $T$ such that

$$T_\ell \rightharpoonup T \text{ as } \ell \to \infty, \qquad \text{and } \mathbf{M}_W(T) < \liminf_{\ell\to\infty} \mathbf{M}_W(T_\ell).$$

The following useful result is more or less a direct consequence of our earlier criteria for weak compactness of sequences of measures.

**Proposition 2.** *Assume that* $(T_\ell)$ *is a sequence of currents with locally finite mass in an open set* $U \subset \mathbb{R}^N$*, and that for every compact* $K \subset U$*, there exists some* $C_K$ *such that*

$$\sup_\ell \mathbf{M}_K(T_\ell) \le C_K.$$

*Then there is a subsequence* $(T_{\ell'})$ *and a current* $T$ *such that*

$$T_{\ell'} \rightharpoonup T \qquad \text{as } \ell' \to \infty.$$

Using the results we have proved so far, we can deduce a very weak existence theorem for currents of minimal mass spanning a given boundary.

**Proposition 3.** *Assume that* $T \in \mathcal{D}_k(\mathbb{R}^N)$ *and that there exists some* $S \in \mathcal{D}_{k+1}(\mathbb{R}^N)$ *such that*

$$T = \partial R, \qquad \mathbf{M}(R) < \infty$$

*Then there exists a current* $S \in \mathcal{D}_{k+1}(\mathbb{R}^N)$ *such that*

$$T = \partial S, \qquad \mathbf{M}(S) = \inf\{\mathbf{M}(R) : R \in \mathcal{D}_{k+1}(U), \partial R = T\}.$$

The reason this is a very weak theorem is that it doesn't tell us anything that would make us confident that the minimizing current $S$ has any sort of reasonable geometric structure at all – it is merely a $k+1$-current with finite mass and boundary equal to a given $k$-current $T$.

Proof. Let $S_k$ be a sequence of currents such that

$$\partial S_k = T, \qquad \mathbf{M}(S_k) \to \inf\{\mathbf{M}(R) : R \in \mathcal{D}_{k+1}(U), \partial R = T\} < \infty$$

as $k \to \infty$.

Then it follows from Proposition 2 that there is a subsequence $k'$ that converges weakly to a limit $S$.

By definition of weak convergence,

$$\partial S(\omega) = S(d\omega) = \lim_{k'} S_{k'}(d\omega) = \lim_{k'} \partial S_{k'}(\omega) = \lim_{k'} T(\omega) = T(\omega)$$

for all $\omega \in \mathcal{D}^k(\mathbb{R}^N)$. Thus $\partial S = T$.

Also, it follows from (13) that

$$\mathbf{M}(S) \le \liminf_{k'} \mathbf{M}(S_{k'}) = \inf\{\mathbf{M}(R) : R \in \mathcal{D}_{k+1}(U), \partial R = T\}.$$

On the other hand, since $\partial S = T$, it is clear that

$$\mathbf{M}(S) \ge \inf\{\mathbf{M}(R) : R \in \mathcal{D}_{k+1}(U), \partial R = T\}.$$

□

One of our main goals is to prove a similar but stronger theorem, that will assert the existence of a mass-minimizer in the set of *integer multiplicity rectifiable currents* that span a given boundary. Since these currents have a decent, if not exactly *good*, geometric structure, such a theorem will be a much more satisfactory result about existence of minimal surfaces. (It is also a starting-point for further results which show that a minimizing current is a smooth submanifold away from a closed, lower-dimensional set.)

The proof, when we eventually give it, will follow exactly the same easy argument as the proof of Proposition 3. But to give this easy argument, we will need to an analog of Proposition 2 for integer multiplicity rectifiable currents. So we will devote a lot of effort to such a compactness theorem.