**Mat 1501 lecture notes** November 4-8

## 1. the flat norm

The *flat norm* of a current $T \in \mathcal{D}_k(\mathbb{R}^N)$, denoted $\mathbf{F}(T)$, is defined by

$$\mathbf{F}(T) := \sup\{T(\omega) : \omega \in \mathcal{D}^k(\mathbb{R}^N), \max\{\|\omega\|, \|d\omega\|\} \leq 1\} \tag{1}$$

where we recall that $\|\omega\| := \sup(\omega(x), \omega(x))^{1/2}$.

**Remark 1.** One can erify that the flat norm is in fact a norm on the space

$$\{T \in \mathcal{D}_k(\mathbb{R}^N) : \mathbf{F}(T) < \infty\}.$$

The flat norm admits a geometric interpretation:

**Lemma 1.** *If $T \in \mathcal{D}_k(\mathbb{R}^N)$, then*

$$\mathbf{F}(T) = \inf\{\mathbf{M}(T - \partial S) + \mathbf{M}(S) \ : \ S \in \mathcal{D}_{k+1}(\mathbb{R}^N)\}. \tag{2}$$

The proof involves a clever use of the Hahn-Banach Theorem.

Proof. First, if $\omega \in \mathcal{D}^k(\mathbb{R}^N)$ is a $k$-form such that $\max\{\|\omega\|, \|d\omega\|\} \leq 1$, then for any $S \in \mathcal{D}_{k+1}(\mathbb{R}^N)$,

$$T(\omega) = (T - \partial S)(\omega) + \partial S(\omega) = (T - \partial S)(\omega) + S(d\omega) \leq \mathbf{M}(T - \partial S) + \mathbf{M}(S).$$

So we only have to find some $S$ such that equality holds.

To do this, we introduce the space

$$X := \mathcal{D}^k(\mathbb{R}^N) \times \mathcal{D}^{k+1}(\mathbb{R}^N),$$

equipped with the norm

$$\|(\omega, \eta)\|_X := \max\{\|\omega\|, \|\eta\|\}.$$

We also define the linear subspace

$$Y := \{(\omega, \eta) \in X : \eta = d\omega\} = \{(\omega, d\omega) : \omega \in \mathcal{D}^k(\mathbb{R}^N)\}.$$

equipped with the norm inherited from $X$, and a linear functional $L : Y \to \mathbb{R}$:

$$L(\omega, d\omega) = T(\omega).$$

Then the operator norm of $L$ is

$$\begin{aligned} \|L\|_{Y\to\mathbb{R}} &= \sup\{L(\omega, d\omega) : \|(\omega, d\omega)\|_Y \leq 1\} \\ &= \sup\{T(\omega) : \max\{\|\omega\|, \|d\omega\|\} = \mathbf{F}(T). \end{aligned}$$

By the Hahn-Banach Theorem, there exists a linear functional $\bar{L} : X \to \mathbb{R}$ such that $\bar{L}$ agrees with $L$ in $Y$, and whose norm is no larger than that of $L$:

$$\|\bar{L}\|_{X\to\mathbb{R}} = \|L\|_{Y\to\mathbb{R}}.$$

Now we define

$$\begin{aligned} R(\omega) &= \bar{L}(\omega, 0) \quad \text{ for } \omega \in \mathcal{D}^k(\mathbb{R}^N), \\ S(\eta) &= \bar{L}(0, \eta) \quad \text{ for } \eta \in \mathcal{D}^{k+1}(\mathbb{R}^N). \end{aligned}$$

Then $R$ is a $k$-current and $S$ is a $k+1$-current. We claim that

$$T = R + \partial S, \qquad \mathbf{M}(R) + \mathbf{M}(S) = \mathbf{F}(T). \tag{3}$$

Note that this will complete the proof of the lemma. To prove (3), first note that since $L = \bar{L}$ in $Y$,

$$T(\omega) = \bar{L}(\omega, d\omega) = R(\omega) + S(d\omega) = (R + \partial S)(\omega).$$

Also,

$$\begin{aligned}\mathbf{F}(T) &= \|\bar{L}\|_{X\to\mathbb{R}} \\ &= \sup\{\bar{L}(\omega,\eta) : \|(\omega,\eta)\|_X \le 1\} \\ &= \sup\{R(\omega) + S(\eta) : \|(\omega\| \le 1, \|\eta \le 1\} \\ &= \mathbf{M}(R) + \mathbf{M}(S).\end{aligned}$$

□

A variant that is sometimes useful is the *homogeneous flat norm*, which we will write $\dot{\mathbf{F}}(T)$, defined by

$$\dot{\mathbf{F}}(T) := \sup\{T(\omega) : \omega \in \mathcal{D}^k(\mathbb{R}^N),\ \|d\omega\| \le 1\}. \tag{4}$$

This too admits a geometric interpretation:

**Lemma 2.** *If $T \in \mathcal{D}_k(\mathbb{R}^N)$, then*

$$\dot{\mathbf{F}}(T) = \inf\{\mathbf{M}(S)\ :\ S \in \mathcal{D}_{k+1}(\mathbb{R}^N), \partial S = T\}. \tag{5}$$

In particular, (5) implies that $\dot{\mathbf{F}}(T) = +\infty$ unless $T = \partial S$ for some $S$.

**Exercise 1.** Prove Lemma 2.

## 2. products and homotopy of currents

Next we introduce a couple of useful constructions that will enable us to prove, for example, that if $T \in \mathcal{D}_k(\mathbb{R}^N)$ is a current such that $\partial T = 0$, then there exists some $S \in \mathcal{D}_{k+1}(\mathbb{R}^N)$ such that $\partial S = T$.

**2.1. the product of currents.** First, we define the product of currents $S \in \mathcal{D}_k(U)$ and $T \in \mathcal{D}_\ell(V)$. Let us assume that $(x_1, \ldots, x_N)$ are coordinates on $U$, and $(y_1, \ldots, y_M)$ are coordinates on $V$. A smooth $k+\ell$-form $\omega$ on $U \times V$ can be written

$$\omega = \sum_{\substack{k',\ell' \ge 0,\\ k'+\ell' = k+\ell}} \sum_{\alpha \in I(N,k')} \sum_{\beta \in I(M,\ell')} a_{\alpha\beta}(x,y) dx^\alpha \wedge dy^\beta$$

where $a_{\alpha\beta}$ is a smooth function on $U \times V$. For $\omega$ as above, we define

$$(S \times T)(\omega) = S\left(\sum_{\alpha \in I(N,k)} T\left(\sum_{\beta \in I(M,\ell)} a_{\alpha\beta}(x,y) dy^\beta\right) dx^\alpha\right).$$

**Exercise 2.** Check that for $S, T$ as above,

$$\partial(S \times T) = \partial S \times T + (-1)^k T \times \partial S.$$

**Example 1.** Suppose that $T \in \mathcal{D}_k(U)$, and let $[\![(0,1)]\!]$ denote current associated to the interval $(0,1) \subset \mathbb{R}$.

Then

$$\partial([\![(0,1)]\!] \times T) = ([\![1]\!] - [\![0]\!]) \times T - [\![(0,1)]\!] \times \partial T.$$

### 2.2. the homotopy formula.

**Lemma 3.** *Assume that $U, V$ are subsets of Euclidean spaces, and that $h : [0,1] \times U \to V$ is a smooth map.*

*Let $f(x) = h(x,0)$ and $g(x) = h(x,1)$, so that $h$ may be described as a homotopy between $f$ and $g$.*

*If $T \in \mathcal{D}_k(U)$ and the restriction of $h$ to supp$([\![(0,1)]\!] \times T)$, is proper, then*

$$\boxed{g_*T - f_*T = h_*([\![(0,1)]\!] \times \partial T) - \partial h_*([\![(0,1)]\!] \times T).} \tag{6}$$

Equation (6) is known as *the homotpoy formula*

Proof. First note that the hypothesis imply that all the currents appearing in (6) are well-defined.

To prove the formula, we simply compute

$$\begin{aligned}\partial h_*([\![(0,1)]\!] \times T) &= h_*\partial([\![(0,1)]\!] \times T) \\ &= h_*\big([\![1]\!] \times T - [\![0]\!] \times T - [\![(0,1)]\!] \times \partial T\big) \\ &= g_*T - f_*T - h_*([\![(0,1)]\!] \times \partial T)\end{aligned}$$

We deduce (6) by rearranging this. □

**Exercise 3.** Verify in detail the fact, already used in the above proof, that under the hypotheses of the above lemma,

$$h_*\big([\![1]\!] \times T\big) = g_*T.$$

(The proof that $h_*\big([\![0]\!] \times T\big) = f_*T$ is identical.)

Since the right-hand side of the homotopy formula has the form $R + \partial S$, it is natural to use it to bound the distance in the flat norm between $f_*T$ and $g_*T$.

**Lemma 4.** *Assume that $T \in \mathcal{D}_k(U)$ is a current such that $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$, and that $h : [0,1] \times U \to V$ is smooth, with $Lip(h) < \infty$. Let $f(x) = h(0,x)$ and $g(x) = h(1,x)$ as above. Then*

$$\begin{aligned}\mathbf{F}(f_*T - g_*T) \le &\sup_{(t,x)\in[0,1]\times supp(T)} \left(|\frac{\partial h}{\partial t}|\ |\nabla_x h|^k\right)\mathbf{M}(T) \\ &+ \sup_{(t,x)\in[0,1]\times supp(T)} \left(|\frac{\partial h}{\partial t}|\ |\nabla_x h|^{k-1}\right)\mathbf{M}(\partial T)\end{aligned} \tag{7}$$

*where $|\nabla_x h| := \sup_{|v|\le 1} |(\nabla_x h)v|$.*

This will most often be applied to the function $h(t,x) = tg(x) + (1-t)f(x)$, which is known as the *affine homotopy* between $f$ and $g$. In this case, $|\partial_t h(t,x)| = |f(x) - g(x)|$.

Proof. The homotopy formula and Lemma 1 imply that

$$\mathbf{F}(f_*T - g_*T) \le \mathbf{M}(R) + \mathbf{M}(S),$$

for

$$R := h_*([\![(0,1)]\!] \times \partial T) \qquad S := h_*([\![(0,1)]\!] \times T).$$

So we just have to bound $\mathbf{M}(R), \mathbf{M}(S)$ in terms of $\mathbf{M}(\partial T), \mathbf{M}(T)$, together with properties of $Lip(h)$. To do this, we explicitly represent $R, S$ as integration against certain measures.

Considering first $S$ (in fact the argument for $R$ will be essentially identical) we first recall that $T$ can be represented by integration:

$$T(\omega) = \int_U \langle \omega(x), \vec{T}(x) \; d\mu_T(x).$$

where $|\vec{T}| = 1$, $\mu_T$ *a.e.*. In this case, we can write

$$[\![(0,1)]\!] \times T(\eta) = \int_{[0,1]\times U} \langle \eta, e_t \wedge \vec{T} \rangle d(\mathcal{L}^1 \times \mu_T) \qquad \text{for } \eta \in \mathcal{D}^{k+1}(\mathbb{R} \times U)$$

where $e_t$ denotes the positively oriented unit vector that spans $\mathbb{R}$. (This can be verified from the definitions.) It follows that for $\eta \in \mathcal{D}^{k+1}(V)$,

$$S(\eta) = \big([\![(0,1)]\!] \times T\big)(h^*\eta) = \int_{[0,1]\times U} \langle h^*\eta, e_t \wedge \vec{T} \rangle d(\mathcal{L}^1 \times \mu_T)$$

For $\mathcal{L}^1 \times \mu_T$ a.e. $(t,x)$

$$\langle h^*\eta, e_t \wedge \vec{T} \rangle (t,x) = \langle \eta, h_*(e_t \wedge \vec{T}) \rangle (h(t,x)) = \langle \eta, h_* e_t \wedge h_* \vec{T} \rangle (h(t,x)),$$

and it follows from this that

$$|\langle h^*\eta, e_t \wedge \vec{T} \rangle| \le |\eta \circ h| \; |h_* e_t| \; |h_* \vec{T}|.$$

and one can check that

$$|h_* e_t| = |\frac{\partial h}{\partial t}|, \qquad |h_* \vec{T}| \le |\nabla_x h|^k \tag{8}$$

It follows that if $\|\eta\| \le 1$, then

$$\begin{aligned} |S(\eta)| \;&\le\; \int_{[0,1]\times U} |\frac{\partial h}{\partial t}| \; |\nabla_x h|^k d(\mathcal{L}^1 \times \mu_T) \\ &\le \sup_{(t,x)\in[0,1]\times \mathrm{supp(T)}} \left( |\frac{\partial h}{\partial t}| \; |\nabla_x h|^k \right) \mathbf{M}(T). \end{aligned}$$

By taking the supremum over $\eta$ such that $\|\eta\| \le 1$, we find that

$$\mathbf{M}\,(S) \;\le \sup_{(t,x)\in[0,1]\times \mathrm{supp(T)}} \left( |\frac{\partial h}{\partial t}| \; |\nabla_x h|^k \right) \mathbf{M}(T).$$

Exactly the same considerations apply to $R$ (with $k$ replaced by $k-1$), so we deduce (7). ☐

**Exercise 4.** Verify that (8) holds.

### 2.3. applications of the homotopy formula.

**Proposition 1.** *Assume that $T \in \mathcal{D}_k(\mathbb{R}^N)$ for some $k \ge 1$, and that $\partial T = 0$. Then there exists $S \in \mathcal{D}_{k+1}(\mathbb{R}^N)$ such that*

$$T = \partial S$$

Proof. Let $h(t,x) = tx$, and let $S = -h_*([\![(0,1)]\!] \times T)$ Then

- $g(x) = h(1,x) = x$, so that $g_* T = T$.
- $f(x) = h(0,x) = 0$, so that $f_* T = 0$.
- Since $\partial T = 0$, clearly $h_*([\![(0,1)]\!] \times \partial T) = 0$.

So the homotopy formula implies that $\partial S = T$. ☐

**Exercise 5.** Verify that the above proposition is not true for $k = 0$. Where does the proof go wrong?

Of course the proof is still valid if $T \in \mathcal{D}_n(U)$ for some star-shaped $U \subset \mathbb{R}^N$, and more generally if $U$ is contractible in the sense that there exists some smooth $h : [0,1] \times U \to U$ such that $h(0,x)$ is constant and $h(1,x) = x$.

Another use of the homotopy formula is that it allows us to extend the definition of the push-forward of a current. For example, the following result says, in effect, that under suitable hypotheses, to define $f_*T$, we only need to know the behavior of $f$ on $\mathrm{supp}(T)$.

**Lemma 5.** *Assume that $T \in \mathcal{D}_k(U)$ and that $T$ and $\partial T$ have locally finite mass in $U$.*

*If $f, g : U \to V$ are smooth and $f = g$ on supp$(T)$, and if moreover the restriction of $f$ (and hence of $g$) to supp$(T)$ is proper, then*

$$f_*T = g_*T$$

Proof. The $h(t,x) = tg(x) + (1-t)f(x)$ be the natural affine homotopy between $f$ and $g$. Then our hypotheses imply that $\partial_t h = 0$ on the support of $T$, so it follows from (7) that

$$\mathbf{F}(f_*T - g_*T) = 0.$$

□

The next result, in a similar spirit, allows us to define $f_*T$ if $f$ is merely Lipschitz, again subject to some additional natural assumptions. The strategy is to approximate $f$ by smooth functions. Toward this end, we will write $\psi_\varepsilon$ to denote a nonnegative function of the form

$$\psi_\varepsilon(x) = \frac{1}{\varepsilon^N}\psi(\frac{x}{\varepsilon}), \qquad \text{where} \quad \psi \in C_c^\infty(\mathbb{R}^N), \quad \mathrm{supp}\psi \subset B(0,1), \quad \int \psi = 1.$$

**Lemma 6.** *Assume that $T \in \mathcal{D}_k(U)$ and that $T$ and $\partial T$ have locally finite mass in $U$.*

*Assume that $f : U \to V$ is Lipschitz and that the restriction of $f$ to supp$(T)$ is proper.*

*Let $\psi \in C_c^\infty(\mathbb{R}^N)$ be a fixed nonnegative function such that supp$(\psi) \subset B(0,1)$ and $\int \psi = 1$, and let*

$$\psi_\varepsilon(x) := \frac{1}{\varepsilon^N}\psi(\frac{x}{\varepsilon}), \qquad f_\varepsilon := \psi_\varepsilon * f.$$

*Then*

$$f_*T := \lim_{\varepsilon \to o} f_{\varepsilon *}T \qquad \textit{exists,}$$

*and*

$$(9) \qquad supp(f_*T) \subset f(supp(T)), \qquad \mathbf{M}_W(f_*T) \le (Lip(f))^k \mathbf{M}_{f^{-1}(W)}(T)$$

*for all open $W$ such that $\bar{W}$ is a compact subset of $V$.*

sketch of proof. Fix $\varepsilon, \varepsilon' > 0$ and let $h$ be the affine homotopy between $f_\varepsilon$ and $f_{\varepsilon'}$, that is

$$h(t,x) := tf_\varepsilon(x) + (1-t)f_{\varepsilon'}(x).$$

One can check (this is an exercise, see below for some hints) that

$$(10) \qquad |h_t(x)| = |f_\varepsilon(x) - f_{\varepsilon'}(x) \le \mathrm{Lip}(f)(\varepsilon + \varepsilon'),$$

and

$$|\nabla_x h(x)| \le \mathrm{Lip}(f) \tag{11}$$

for all $x$. It then follows from Lemma 4 that for any $\omega \in \mathcal{D}^k(U)$,

$$|f_{\varepsilon *}T(\omega) - f_{\varepsilon' *}T(\omega)| \le (\varepsilon + \varepsilon')(\mathbf{M}(T) + \mathbf{M}(\partial T))\mathrm{Lip}(f)^{k+1}(\max \|\omega\|, \|d\omega\|)$$

It follows that $\lim_{\varepsilon\to 0} f_{\varepsilon *}T(\omega)$ exists. Both claims in (9) are proved by first considering smooth $f_\varepsilon$ and then letting $\varepsilon \to 0$ (necessarily, since this is how we define $f_*T$.) For the inequality involving $\mathbf{M}_W(f_*T)$, it suffices to check that

$$\mathbf{M}_{W'}(f_*T) \le (\mathrm{Lip}(f))^k \mathbf{M}_{f^{-1}(W)}(T)$$

for any open $W'$ such that $\bar{W}$ is a compact subset of $W$, and this is easier than working directly with $W$ on the left-hand side. We omit the details. $\square$

**Exercise 6.** Check that (10), (11) hold.

To prove (10) since $|f_\varepsilon - f_{\varepsilon'}| \le |f_\varepsilon - f| + |f - f_{\varepsilon'}|$, it suffices to check that $|f_\varepsilon(x) - f(x)| \le \mathrm{Lip}(f)\varepsilon$.

For the proof of (11), it may be helpful to note that

$$h(t,x) := (t\psi_\varepsilon + (1-t)\psi_{\varepsilon'}) * f.$$

and to recall elementary properties of convolution, including Young's inequality.

## 3. Some theorems and some examples

Here are some central results about currents. We will prove as much of these as we can in what is left of this term.

**Theorem 1** (Closure theorem)**.** *Let $T_j$ be a sequence of integer multiplicity rectifiable $k$-currents in $R^N$ such that*

$$\sup_j \big(\mathbf{M}_W(T_j) + \mathbf{M}_W(\partial T_j)\big) < \infty \qquad \text{for every bounded open } W \subset \mathbb{R}^N. \tag{12}$$

*Then there is a subsequence $j'$ and an* integer multiplicity rectifiable $T \in \mathcal{D}_k(\mathbb{R}^N)$ *such that*

$$T_{j'} \rightharpoonup T \qquad \text{as } j' \to \infty.$$

**Theorem 2** (Boundary rectifiability theorem)**.** *Assume that $T$ is an integer multiplicity rectifiable $k$-current in $\mathbb{R}^N$ for $k \ge 1$ . If $\mathbf{M}(\partial T) < \infty$, then $\partial T$ is integer multiplicity rectifiable.*

To illustrate the content of the Closure Theorem we give a number of examples illustrating ways in which, if the hypothesis are not satisfied, a sequence of i.m. rectifiable currents may converge to a current that fails to be rectifiable.

In all the examples, we will use the notation: for $p = (p_1, p_2) \in \mathbb{R}^2$ and $r > 0$,

$$I(p,r) := [p_1, p_1 + r] \times \{p_2\}$$

Thus $I(p,r)$ is an interval of length $r$, parallel to the $x$ axis and with its left endpoint at $p$.

We will always take $I(p,r)$ to be oriented by the tangent vector $e_1 = (1,0)$.

We will always write a generic 1-form $\omega$ as $\omega = \omega_1 dx^1 + \omega_2 dx^2$.

Then of course $\langle \omega, e_1\rangle = \omega_1$.

**Example 2.** Let $T_j = j[\![I(0,\frac{1}{j})]\!]$, so that

$$T_j(\omega) = j\int_{I(0,\frac{1}{j})} \omega_1 d\mathcal{H}^1 = j\int_0^{\frac{1}{j}} \omega_1(s,0)\ ds.$$

Then it is easy to verify that $T_j \rightharpoonup T$ as X$j\to\infty$, for

$$T(\omega) = \omega_1(0).$$

This is not a 1-dimensional i.m. rect current because it is too concentrated – it involves integration over a 0-dimensional set rather than a 1-d set. (It is also of course not a i.m. rect 0-current, because it is not a 0-current – it is a linear functional on 1-forms rather than 0-forms.)

**Example 3.** Let

$$T_j = \sum_{0\le p_1,p_2\le j-1} [\![(I(\frac{p}{k},\frac{1}{j^2})]\!], \qquad\qquad p=(p_1,p_2).$$

Then

$$T_j(\omega) = \sum_{0\le p_1,p_2\le j-1} \int_{I(\frac{p}{k},\frac{1}{j^2})} \omega_1\ d\mathcal{H}^1,$$

and from this one can check that

$$T_j \rightharpoonup T, \qquad T(\omega) = \int_0^1\int_0^1 \omega_1(x_1,x_2)dx_1\,dx_2.$$

This is not a 1-dimensional i.m. rect current because it is too spread out – it involves integration over a 2-dimensional set rather than a 1-d set.

**Example 4.** Let

$$T_j := \sum_{0\le p\le j-1} [\![I((0,\frac{p}{j}),\frac{1}{j})]\!]$$

Then

$$T_j(\omega) = \sum_{p=0}^{j-1}\int_{I((0,\frac{p}{j}),\frac{1}{j})} \omega d\mathcal{H}^1$$

from which one can check that

$$T_j \rightharpoonup T, \qquad T(\omega) = \int_0^1 \omega_1(0,y)dy = \int_{\{0\}\times[0,1]} \langle\omega,e_1\rangle d\mathcal{H}^1.$$

This is not a rectifiable current, since the vector $e_1$ is nowhere tangent to the set $\{0\}\times[0,1]$ over which we integrate.

## 4. slicing

**4.1. introduction to slicing.** Many of our arguments, including the proofs of Theorems 1 and 1, will rely on the notion of slicing. Before giving a complete treatment of it, we first give a slightly formal discussion, in which we will attempt to convey some main ideas without supplying all technical details.

Assume that $T$ is a $k$-dimensional current in $\mathbb{R}^N$, and let $f:\mathbb{R}^N\to\mathbb{R}^n$ be a Lipschitz function, with $0<n\le k$.

Under suitable hypotheses on $T$, we will define the *slices of $T$ by level sets of $f$*, which will be $k-n$-currents, denoted

$$\langle T,f,y\rangle \qquad \text{for } y\in\mathbb{R}^n.$$

We will do this in such a way that if $M$ is a smooth oriented $k$-dimensional submanifold and $f$ is smooth, then

$$\langle [\![M]\!], f, y\rangle \;=\; [\![M \cap f^{-1}\{y\}]\!] \qquad \text{for } a.e.\ y \in \mathbb{R}^n, \tag{13}$$

where, to make sense of the right-hand side, we will need to have some way of determining an orientation of $M \cap f^{-1}(y)$ from the orientation of $M$ and properties of $f$.

The most important example occurs when $f$ is just projection onto an $n$-dimensional subspace, in which case $f^{-1}\{y\}$ is a codimension $n$ plane for every $y$.

The definition we give is slightly opaque: we will require that for $y \in \mathbb{R}^n$, the slices $\langle T, f, y\rangle$ satisfy

$$\operatorname{supp}(\langle T, f, y\rangle) \subset f^{-1}\{y\} \tag{14}$$

and

$$\int_{\mathbb{R}^n} \langle T, f, y\rangle(\phi)\ \eta = T(f^*\eta \wedge \phi) \qquad \text{for every } \phi \in \mathcal{D}^{k-n}(\mathbb{R}^N) \text{ and } \eta \in \mathcal{D}^n(\mathbb{R}^n). \tag{15}$$

We will prove later on that such slices exist, and are almost uniquely determined for $\mathcal{L}^n$ a.e.$y \in \mathbb{R}^n$.

First, to attempt to motivate condition (15), we show in the following lemma that it is consistent with (13) in an easy special case.

**Lemma 7.** *Let $A$ be a bounded, open subset of $\mathbb{R}^N$ with smooth boundary. Define $\pi : \mathbb{R}^n \to \mathbb{R}^n$ by*

$$\pi(x_1, \ldots, x_N) = (x_1, \ldots, x_n),$$

*and for $y \in \mathbb{R}^n$, let*

$$A_y := A \cap \pi^{-1}\{y\}$$

*oriented by the tangent multivector $\tau := e_{n+1} \wedge \ldots \wedge e_N$. Then for every $\phi \in \mathcal{D}^{N-n}(\mathbb{R}^N)$ and every $\eta \in \mathcal{D}^n(\mathbb{R}^n)$,*

$$[\![A]\!](\pi^*\eta \wedge \phi) \;=\; \int_{y\in\mathbb{R}^n} [\![A_y]\!](\phi)\ \eta \tag{16}$$

*Similarly, for every $\phi \in \mathcal{D}^{N-n-1}(\mathbb{R}^N)$ and every $\eta \in \mathcal{D}^n(\mathbb{R}^n)$,*

$$[\![\partial A]\!](\pi^*\eta \wedge \phi) \;=\; (-1)^n \int_{y\in\mathbb{R}^n} [\![\partial A_y]\!](\phi)\ \eta \tag{17}$$

**Remark 2.** On the right-hand side of (16), $y \mapsto [\![A_y]\!](\phi)$ is a function of $y$ which we will see is measurable, and $\eta$ is an $n$-form. So their product is again an $n$-form, and hence something that we know how to integrate over $\mathbb{R}^n$. On the right-hand side, $\pi^*\eta$ is a $n$-form and $\phi$ a $k-n$-form, so $\pi^*\eta \wedge \phi$ is a $k$-form, and $[\![A]\!](\pi^*\eta \wedge \phi)$ makes sense. Similar considerations of course apply to (17).

**Remark 3.** Comparing the definition of slices (14), (15), we see that the lemma shows that

$$\langle [\![A]\!], \pi, y\rangle = [\![A_y]\!], \qquad \langle [\![\partial A]\!], \pi, y\rangle = (-1)^n [\![\partial A_y]\!]$$

for a.e. $y$.

PROOF. Fix $\phi \in \mathcal{D}^{k-n}(\mathbb{R}^N)$ and $\eta \in \mathcal{D}^n(\mathbb{R}^n)$. We can write $\eta$ in the form $\eta = \eta_0(y)dy$, where $\eta_0(y)$ is a smooth, compactly supported function and $dy = dy^1 \wedge \ldots \wedge dy^n$. Then

$$\pi^*\eta(x) = \eta_0(x_1, \ldots, x_n)dx^1 \wedge \ldots \wedge dx^n = \eta(\pi(x))dx^1 \wedge \ldots \wedge dx^n.$$

It follows that

$$\pi^*\eta \wedge \phi = \eta \circ \pi \ \langle \phi, \tau \rangle dx^1 \wedge \ldots \wedge dx^N, \tag{18}$$

as one can check by writing both sides out in components. Then by Fubini's Theorem,

$$\begin{aligned}
[\![A]\!](\pi^*\eta \wedge \phi) &= \int_A \eta \circ \pi(x) \ \langle \phi, e_{\alpha_*} \rangle d\mathcal{H}^N \\
&= \int_{\mathbb{R}^n} \eta(y) \left( \int_{A_y} \langle \phi, e_{\alpha_*} \rangle d\mathcal{H}^{N-n} \right) \ d\mathcal{L}^n(y) \\
&= \int_{\mathbb{R}^n} [\![A_y]\!](\phi) \ \eta(y) \ d\mathcal{L}^n(y).
\end{aligned}$$

This proves (16). To prove (17), we note that $d\pi^*\eta = \pi^* d\eta = 0$, since $d\eta$ is a $n+1$-form on $\mathbb{R}^n$, and hence necessarily vanishes. Thus (16) implies that

$$\begin{aligned}
\partial[\![A]\!](\pi^*\eta \wedge \phi) &:= [\![A]\!](d(\pi^*\eta \wedge \phi)) \\
&= (-1)^n [\![A]\!](\pi^*\eta \wedge d\phi)) \\
&= (-1)^n \int_{\mathbb{R}^n} [\![A_y]\!](d\phi) \ \eta(y) \ d\mathcal{L}^n(y) \\
&= (-1)^n \int_{\mathbb{R}^n} \partial[\![A_y]\!](\phi) \ \eta(y) \ d\mathcal{L}^n(y).
\end{aligned}$$

Also, for a.e. $y$, our hypotheses imply that $A_y$ is a $N-1$-dimensional submanifold of $\mathbb{R}^N$ with smooth boundary, and for such $y$, Stokes' Theorem implies that $\partial[\![A_y]\!] = [\![\partial A_y]\!]$. □

Next, we prove a more general version of the above lemma, in which we consider slicing by level sets of an arbitrary Lipschitz function, rather than a projection.

The outline of the computation is similar, with Fubini's Theorem replaced by the coarea formula. A new technical point is that specifying the orientation of the slices becomes more complicated – compare (18) above, which is a straightforward verification, with (20) below, the proof of which requires a certain amount of multilinear algebra.

**Lemma 8.** *Let $A$ be a bounded, open subset of $\mathbb{R}^N$ with smooth boundary.*

*Let $f : \mathbb{R}^N \to \mathbb{R}^n$ be Lipschitz, and for $y \in \mathbb{R}^n$, let*

$$A_y := A \cap f^{-1}\{y\}.$$

*Then $A_y$ is also locally $N-n$-rectifiable for $\mathcal{L}^n$ a.e. $y \in \mathbb{R}^n$, and can be oriented in such a way that*

$$[\![A]\!](f^*\eta \wedge \phi) \ = \ \int_{y \in \mathbb{R}^n} [\![A_y]\!](\phi) \ \eta. \tag{19}$$

Proof. **1**. Define

$$\begin{aligned} A^+ &:= \{x \in A : f \text{ is differentiable at } x, \text{ and } Jf(x) > 0\},\\ A^0 &:= A \setminus A^+,\\ A^+_y &:= A^+ \cap f^{-1}\{y\},\\ A^0_y &:= A^0 \cap f^{-1}\{y\}. \end{aligned}$$

The definition of differentiability implies that the kernel of $\nabla f(x)$ is an $N-n$-dimensional approximate tangent plane for $A^+_{f(x)}$ at every point in $A^+$. Thus an $N-n$-dimensional approximate tangent plane exists at every point of $A^+_y$, for every $y \in \mathbb{R}^n$. Also, it follows from the coarea formula that $\mathcal{H}^{N-n}(A^0_y) = 0$ and that $\mathcal{H}^{N-n}(A^+_y) < \infty$ for $\mathcal{L}^n$ a.e. $y$. These facts establish that $A_y$ is locally $N-n$-rectifiable for $\mathcal{L}^n$ a.e. $y \in \mathbb{R}^n$.

**2**. We next claim that there exists a measurable function $\tau : A^+ \to \Lambda_{N-n}\mathbb{R}^N$ such that $\tau(x)$ orients $T_x A_{f(x)}$, and

$$f^*\eta \wedge \phi = \eta_0 \circ f \ Jf \ \langle \phi, \tau\rangle dx^1 \wedge \ldots \wedge dx^N, \tag{20}$$

in $A^+$, for every $\phi \in \mathcal{D}^{N-n}(\mathbb{R}^N)$ and every $\eta = \eta_0(y)dy \in \mathcal{D}^n(\mathbb{R}^n)$. (Here and below, we use the notation $dy = dy^1 \wedge \ldots \wedge dy^n$.)

To prove this, we note that at every $x \in A^+$,

$$f^*\eta = \eta_0 \circ f \ df^1 \wedge \ldots \wedge df^n = \eta_0 \circ f \ \frac{df^1 \wedge \ldots \wedge df^n}{|df^1 \wedge \ldots \wedge df^n|} Jf$$

So we need to show that there exists a unique $\tau = \tau(x) \in \Lambda_{N-n}\mathbb{R}^N$ that satisfies

$$|\tau| = 1, \qquad \frac{df^1 \wedge \ldots \wedge df^n}{|df^1 \wedge \ldots \wedge df^n|} \wedge \phi = \langle \tau, \phi\rangle dx^1 \wedge \ldots dx^N \quad \text{for all } \phi \in \Lambda^{N-n}(\mathbb{R}^N), \tag{21}$$

and which orients $T_x A_{f(x)} = \ker(\nabla f(x))$.

To see this, fix $x \in A^+$, and fix an orthonormal basis $\{\omega^i\}_{i=1}^N$ for $\Lambda^1 T_x\mathbb{R}^N$ such that

$$\omega^1 \wedge \ldots \wedge \omega^n = \frac{df^1 \wedge \ldots \wedge df^n}{|df^1 \wedge \ldots \wedge df^n|}(x). \tag{22}$$

Let $\{e_i\}_{i=1}^N$ denote the dual basis for $\Lambda_1 T_x\mathbb{R}^N = T_x\mathbb{R}^N$, defined as usual by requiring that

$$\langle \omega^i, e_j\rangle = \delta^i_j. \tag{23}$$

Recall that $\{e_i\}_{i=1}^N$ is also orthonormal. We claim that

$$\tau := e_{n+1} \wedge \ldots \wedge e_N$$

satisfies (21) and orients $T_x A_{f(x)}$. Indeed, it is clear that $|\tau| = 1$, and the other identity in (21) is verified by writing out both sides in terms of the bases $\{\omega^i\}, \{e_i\}$. Finally, it follows from (22) that $\{\omega^1\}_{i=1}^n$ and $\{df^i(x)\}_{i=1}^n$ span the same $n$-plane in $\Lambda_1 T_x\mathbb{R}^N$, and then (23) implies that $\langle df^i(x), e_j\rangle = 0$ for every $i \le n$ and $j > n$. This means that $e_j \in \ker(\nabla f(x))$ for every $j > n$. Since $\ker(\nabla f(x))$ is $(N-n)$-dimensional and $\{e_j\}_{j>n}$ is a set of $N-n$ linearly independent vectors, it follows that $\{e_j\}_{j>n}$ spans $\ker(\nabla f(x))$, and hence that $\tau$ orients $\ker(\nabla f(x))$.

**3**.To conclude the proof, we first remark that at points in $A^0$ where $f$ is differentiable, $Jf(x) = 0$, and hence the set $\{df^i(x)\}_{i=1}^n$ is linearly depdendent, and so

$$f^*\eta = \eta_0 \circ f \; df^1 \wedge \ldots \wedge df^n = 0.$$

Thus

$$\begin{aligned}
[\![A]\!](f^*\eta \wedge \phi) = \int_A f^*\eta \wedge \phi \; &= \int_{A^+} f^*\eta \wedge \phi \\
&\overset{(20)}{=} \int_A \eta_0 \circ f(x) \; Jf(x) \; \langle \phi, \tau \rangle d\mathcal{L}^N(x) \\
&= \int_{\mathbb{R}^n} \eta_0(y) \left( \int_{A_y} \langle \phi, \tau \rangle d\mathcal{H}^{N-n} \right) \; d\mathcal{L}^n(y) \\
&= \int_{\mathbb{R}^n} [\![A_y]\!](\phi) \; \eta.
\end{aligned}$$

This proves (19).

□

**4.2. an interesting calculation.** Assume that $T$ is a 1-current in $\mathbb{R}^N$ such that $\mathbf{M}(T) + \mathbf{M}(\partial T) < \infty$,

Let us consider slices of $T$ by $\pi : \mathbb{R}^N \to \mathbb{R}$ defined by

$$\pi(x_1, \ldots, x_N) = x_1.$$

Assume that $\langle T, \pi, y \rangle \in \mathcal{D}_0(\mathbb{R}^N)$ exist for $\mathcal{L}^1$ a.e. $y \in \mathbb{R}$. (We continue to defer the proof of this, but we will get around to it later.) Rewriting the defining property (15) of slices in this specific setting, we obtain

$$\int_{\mathbb{R}} \langle T, \pi, y \rangle(\phi) \, \eta(y) \, dy = T(\phi \; \eta \circ \pi \, dx^1) \tag{24}$$

for all $\eta \in C_c^\infty(\mathbb{R})$ and $\phi \in \mathcal{D}^0(\mathbb{R}^N) \cong C_c^\infty(\mathbb{R}^N)$. (Note that a wedge product of a function and a 1-form is just ordinary multiplication.)

Let us consider $\eta$ of the form $\eta = \zeta'$ for $\zeta \in C_c^\infty(\mathbb{R})$. Then

$$\begin{aligned}
\phi \; \eta \circ \pi \; dx^1 &= \phi \; \zeta' \circ \pi \; dx^1 \\
&= \phi d(\zeta' \circ \pi) \\
&= d[\phi \; \zeta \circ \pi] - \zeta \circ \pi \; d\phi
\end{aligned}$$

Thus

$$T(\phi \; \eta \circ \pi \, dx^1) = \partial T(\phi \; \zeta \circ \pi) - T(\zeta \circ \pi \; d\phi).$$

Combining this with (24), we deduce that

(25)
$$\left| \int_{\mathbb{R}} \langle T, \pi, y \rangle(\phi) \, \zeta'(y) \, dy \right| \le \max\{\|\phi\|, \|d\phi\|\} \; \left( \int \zeta(x_1) d\mu_T(x) + \int \zeta(x_1) d\mu_{\partial T}(x) \right)$$

for all $\phi \in \mathcal{D}^0(\mathbb{R}^N) \cong C_c^\infty(\mathbb{R}^N)$ and $\zeta \in C_c^\infty(\mathbb{R})$.

By an approximation argument, one can check that (25) remains valid if $\zeta$ is merely Lipschitz, with compact support.

We now claim that for every $a \in \mathbb{R}^n$ and $\phi \in C_c^\infty(\mathbb{R}^N)$,

$$\langle T, \pi, a_- \rangle := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_{a-\varepsilon}^{a} \langle T, \pi, y \rangle \, dy \qquad \text{exists for every } a \in \mathbb{R}. \tag{26}$$

This is an exercise (with hints, see below). If you do the exercise, you will show that the limit may be understood with respect to the flat norm.

Now fix $a < b$, and for $\varepsilon < b - a$, define

$$\zeta_\varepsilon(y) := \begin{cases} 0 & \text{if } y \le a - \varepsilon \text{ or } y \ge b \\ 1 & \text{if } a \le y \le b - \varepsilon \\ \text{linear} & \text{if } a - \varepsilon \le y \le a \text{ or } b - \varepsilon \le y \le b. \end{cases}$$

If we substitute $\zeta_\varepsilon$ in (25), the left-hand side can be written

$$\left| \frac{1}{\varepsilon} \int_{a-\varepsilon}^{a} \langle T, \pi, y\rangle(\phi)\, dy - \frac{1}{\varepsilon} \int_{b-\varepsilon}^{b} \langle T, \pi, y\rangle(\phi)\, dy \right|.$$

Thus, using (25), letting $\varepsilon \to 0$, and using (26), we deduce that

$$\left| \Big( \langle T, \pi, a_-\rangle - \langle T, \pi, b_-\rangle \Big)(\phi) \right| \le \max\{\|\phi\|, \|d\phi\|\} \left( \mathbf{M}_{\pi^{-1}[a,b)}(T) + \mathbf{M}_{\pi^{-1}[a,b)}(\partial T) \right).$$

Equivalently,

$$\mathbf{F}\Big( \langle T, \pi, a_-\rangle - \langle T, \pi, b_-\rangle \Big) \le \mathbf{M}_{\pi^{-1}[a,b)}(T) + \mathbf{M}_{\pi^{-1}[a,b)}(\partial T). \tag{27}$$

Now, for any increasing sequence $\cdots < a_{-1} < a_0 < a_1 < \cdots$ such that $a_j \to \pm\infty$ as $j \to \pm\infty$, by applying (27) on each interval $[a_i, a_{i+1})$ and adding up the results, we find that

$$\sum_{i=-\infty}^{\infty} \mathbf{F}\Big( \langle T, \pi, a_{i-}\rangle - \langle T, \pi, a_{(i+1)-}\rangle \Big) \le \mathbf{M}(T) + \mathbf{M}(\partial T)$$

Since this bound holds for any partition, we conclude that

$y \mapsto \langle T, \pi, y_-\rangle$ **is a function of bounded variation from** $\mathbb{R}$ **into the space** $\{S \in \mathcal{D}_0(\mathbb{R}^N) : \mathbf{F}(S) < \infty\}$ **equipped with the flat norm!**

This fact admits a very natural geometric interpretation, which we will discuss later.

**Exercise 7.** Prove (26).

*longish hint*: This can be done by an argument very similar to the one that was used to deduce (27) from (25) (modulo (26).) The point is that if $0 < \varepsilon' < \varepsilon$, then one can write

$$\frac{1}{\varepsilon} \int_{a-\varepsilon}^{a} \langle T, \pi, y\rangle(\phi)\, dy - \frac{1}{\varepsilon'} \int_{a-\varepsilon'}^{a} \langle T, \pi, y\rangle(\phi)\, dy$$

in the form

$$\int_{\mathbb{R}} \langle T, \pi, y\rangle(\phi)\, \zeta'(y)\, dy$$

for a particular Lipschitz continuous function with support in $[a - \varepsilon, a]$. Then (25) can be used to prove an inequality of the form

$$\mathbf{F}\left( \frac{1}{\varepsilon} \int_{a-\varepsilon}^{a} \langle T, \pi, y\rangle\, dy - \frac{1}{\varepsilon'} \int_{a-\varepsilon'}^{a} \langle T, \pi, y\rangle\, dy \right) \le ???$$

and this can be used to prove (26).