Mat 1501 lecture notes, penultimate installment

1. bounded variation: functions of a single variable (optional)

I believe that we will not actually use the material in this section – the point is mainly to motivate the definition we will later introduce of BV functions $\mathbb{R}^n \to \mathbb{R}$ and to recall some results, perhaps familiar, that we will see extend in a natural way to BV functions of several variables.

(so if you have limited amounts of time, then I recommend that you read other parts of these notes first.)

Definition 1. We will say that a function $f : \mathbb{R} \to \mathbb{R}$ is has *bounded variation* if

$$\sup \sum_{i=-\infty}^{\infty} |f(x_i) - f(x_{i-1})| < \infty, \tag{1}$$

where the supremum is taken over all sequences $\ldots < x_i < x_{i+1} < \ldots$ such that $x_i \to \pm\infty$ as $i \to \pm\infty$. The quantity appearing on the left in (1) is called the *total variation* of f and is denoted $TV(f)$.

Here are some facts about functions of bounded variation that may be familiar from analysis classes:

- Any function of bounded variation may be written as the difference of bounded monotone functions. That is, if f has bounded variation, there exist bounded nondecreasing functions f_1, f_2 such that
$$f = f_1 - f_2. \tag{2}$$
Moreover, this can be done in such a way that
$$TV(f) = TV(f_1) + TV(f_2). \tag{3}$$
- As a result
$$f(x_-) := \lim_{y \nearrow x} f(y) \quad \text{and} \quad f(x_+) := \lim_{y \searrow x} f(y) \quad \text{both exist} \tag{4}$$
for every $x \in \mathbb{R}$.
- If (f^ℓ) is a sequence of functions such that
$$\sup_\ell TV(f^\ell) = M < \infty, \qquad \text{and} \qquad \sup_{\ell, x} |f^\ell(x)| < \infty,$$
then there is a subsequence ℓ' and a function f such that $f^\ell \to f$ a.e. and in L^1, and $TV(f) \le M$.

  Exercise 1. Prove this. One way to do this is to first prove it is true if each f^ℓ is nondecreasing, and then deduce the general case from (2), (3).

- if $f \in C^1$ then
$$TV(f) = \int_{\mathbb{R}} |f'(x)|\, dx, \tag{5}$$
so that a C^1 function has bounded variation if and only if this integral is finite.

  Exercise 2. Prove (5). (In fact it can be deduced in about one sentence from the fundamental theorem of calculus.)

Very closely related to the above definition, we have:

Definition 2. We will say that $f : \mathbb{R} \to \mathbb{R}$ is a BV function if there exists a signed Radon measure μ such that

$$- \int_{\mathbb{R}} f(x)\phi'(x)\ dx = \int_{\mathbb{R}} \phi(x)\, d\mu(x) \qquad \text{for all } \phi \in C^1_c(\mathbb{R}) \tag{6}$$

and

$$|\mu|(\mathbb{R}) < \infty.$$

When (6) holds, we will write $f \in BV(\mathbb{R})$, and we will write f' to denote the signed measure μ in (6).

Note that if f is C^1, then

$$- \int_{\mathbb{R}} f(x)\phi'(x)\ dx = \int_{\mathbb{R}} \phi(x) f'(x)\ dx \qquad \text{for all } \phi \in C^1_c(\mathbb{R}).$$

Motivated by this, we interpret (6) as asserting that the (weak) derivative of f is a signed measure. Hence the notation f' for the measure in (6).

The relationship between the two definitions is given by

Lemma 1. *If $f : \mathbb{R} \to \mathbb{R}$ has bounded variation, then $f \in BV(\mathbb{R})$.*

Moreover, for any interval (a, b), the measure f' satisfies

$$u'((a, b)) = f(b_-) - f(a_+). \tag{7}$$

Conversely, if $f \in BV(\mathbb{R})$, then there exists a function $\tilde{f}$ of bounded variation such that $\tilde{f} = f$ a.e..

Proof. **1**. First, we will prove (6) under the assumption that f is a bounded, nondecreasing function.

Assume that this holds, and define $\lambda : C^1_c(\mathbb{R}) \to \mathbb{R}$

$$\lambda(\phi) := - \int_{\mathbb{R}} f(x)\phi'(x)\ dx$$

By the dominated convergence theorem and a change of variables,

$$\lambda(\phi) = - \lim_{h \to 0} \int f(x) \frac{\phi(x-h) - \phi(x)}{h}\ dx = \lim_{h \to 0} \int \frac{f(x) - f(x-h)}{h} \phi(x)\, dx.$$

The fact that f is nondecreasing implies that

$$\lambda(\phi) \geq 0 \qquad \text{if } \phi \geq 0. \tag{8}$$

Hence $\lambda(\phi) \leq \lambda(\psi)$ if $\phi(x) \leq \psi(x)$ for all x.

In particular, if $\|\phi\|_{sup} \leq 1$ and $\operatorname{supp}(\phi) \subset (a, b)$, then

$$\begin{aligned}
\lambda(\phi) &\leq \lim_{h \searrow 0} \int_{\mathbb{R}} \frac{f(x) - f(x-h)}{h}\ \mathbf{1}_{(a,b)}(x)\ dx \\
&= \lim_{h \searrow 0} \frac{1}{h} \left(\int_{a-h}^{a} f(x)\, dx + \int_{b-h}^{b} f(x)\, dx \right) \\
&= f(b_-) - f(a_-).
\end{aligned}$$

It follows that λ is continuous with respect to the topology of $C_c(\mathbb{R})$. Since its domain of definition is a dense subset of $C_c(\mathbb{R})$, we conclude that λ has a unique extension to a continuous linear functional $\bar{\lambda} : C_c(\mathbb{R}) \to \mathbb{R}$. Such a functional can be represented by a Radon measure μ. (since λ is positive in the sense of (8), and clearly $\bar{\lambda}$ inherits this property.) In particular, (6) holds for this μ.

2. If f has bounded variation, we can write it as a difference of nondecreasing functions $f_1 - f_2$ and apply Step 1 to both of these, to obtain a measure μ satisfying (6).

3. We remark that if f has bounded variation, then (6) continues to hold if ϕ is merely Lipschitz continuous. This can be deduced by approximating ϕ by smooth functions in a standard way and applying (6) (in the smooth case) to each approximant.

Now for $a < b$ and $0 < \varepsilon < \frac{1}{2}(b-a)$, let

$$\phi_\varepsilon(x) := \begin{cases} 0 & \text{if } x \notin (a,b) \\ 1 & \text{if } x \in (a+\varepsilon, b-\varepsilon) \\ \text{linear} & \text{if } x \in [a, a+\varepsilon] \cup [b-\varepsilon, b]. \end{cases}$$

Then using (6),

$$\begin{aligned} \mu((a,b)) &= \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} \phi_\varepsilon(x)\, d\mu(x) \\ &= \lim_{\varepsilon \searrow 0} - \int_{\mathbb{R}} f(x)\phi_\varepsilon'(x) \\ &= \lim_{\varepsilon \searrow 0} \left(-\frac{1}{\varepsilon} \int_a^{a+\varepsilon} f(x)\, dx + \frac{1}{\varepsilon} \int_{b-\varepsilon}^b f(x)\, dx \right) \\ &= f(b_-) - f(a_+). \end{aligned}$$

Hence (7) holds.

4 Now assume that $f \in BV(\mathbb{R})$.

Without giving all the details, there are a couple of ways to show that f coincides a.e. with a function $\tilde{f}$ on bounded variation.

Let us define $g(x)$ by

$$g(x) := \mu((-\infty, x))$$

One can then check that g has bounded variation, and that $(f-g)' = 0$ in the sense that

$$\int (f-g)\phi' \; dx = 0 \qquad \text{for all } \phi \in C^1_c(\mathbb{R}).$$

One can further check that, as a consequence, there exists some constant c such that $f - g = c$ a.e.. Then the conclusions follow, with $\tilde{f} = g + c$.

Alternatively, one can write $f_\varepsilon := \psi_\varepsilon * f$, where ψ_ε is a standard mollifier. Then for every $e > 0$, one can check that

$$TV(f_\varepsilon) = \int_{\mathbb{R}} |f_\varepsilon'(x)|\, dx \le |\mu|(\mathbb{R}).$$

Then it follows from Exercise 1 that there exists a subsequence that converges a.e. to a function $\tilde{f}$ of bounded variation. On the other hand, we know that $f_\varepsilon \to f$ in L^1, and hence (upon passing to a subsequence) almost everywhere. Thus $f = \tilde{f}$ a.e..

$\square$

We will be interested in generalizations of the above definitions to spaces of functions $\mathbb{R}^N \to \mathbb{R}^n$, for example, or more generally $\mathbb{R}^N \to E$, where E is a metric space.

2. bounded variation : functions of several variables

We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is a BV function if $u \in L^1(\mathbb{R}^n)$, and there exists some $C > 0$ such that

$$\int_{\mathbb{R}^n} u(y)\ \nabla \cdot \zeta(y)\, dy \le C\|\zeta\|_\infty \qquad \text{for all } \zeta \in C^1_c(\mathbb{R}^n, \mathbb{R}^n).$$

If this holds, we will write $u \in BV(\mathbb{R}^N)$. For such a function u, the map

$$\zeta \in C^1_c(\mathbb{R}^n;\mathbb{R}^n) \mapsto -\int_{\mathbb{R}^n} u(y)\ \nabla \cdot \zeta(y)\, dy$$

is a linear functional that is continuous with respect to the topology of $C_0(\mathbb{R}^n, \mathbb{R}^n)$ and hence extends to a bounded linear functional $\lambda_u \in C_0(\mathbb{R}^n, \mathbb{R}^n)^*$.

Then a version of the Riesz Representation Theorem asserts that there exists a Radon measure on $\mathbb{R}^n$, which we will denote $|Du|$, and a $|Du|$-measurable function $\sigma : \mathbb{R}^n \to \mathbb{R}^n$, such that

$$\lambda_u(\zeta) = \int_{\mathbb{R}^n} \zeta \cdot \sigma\ d|Du| \qquad \text{for all } \zeta \in C_0(\mathbb{R}^n, \mathbb{R}^n)$$

We will also sometimes write u_{x_i} to denote the signed measure defined by

$$u_{x_i}(A) = \int_A \sigma^i\ d|Du|, \qquad A \text{ Borel}$$

and Du to denote the vector-valued measure $(u_{x_1}, \ldots, u_{x_n}) = |Du| \llcorner \sigma$. Then we can identify Du as the gradient of u in the weak sense.

For this reason, one often says that $u \in BV(\mathbb{R}^n)$ *if and only if Du is a measure.*

We recall that the Riesz Representation Theorem also guarantees that

$$|Du|(O) = \sup\left\{\int_{\mathbb{R}^n} u\ \nabla \cdot \zeta\, d\mathcal{L}^n : \zeta \in C^1_c(\mathbb{R}^n;\mathbb{R}^n), \mathrm{supp}(\zeta) \subset O, \|\zeta\|_\infty \le 1\right\}$$

for any open $O \subset \mathbb{R}^n$.

Note that Du, σ and $|Du|$ are characterized by the identity

$$-\int_{\mathbb{R}^n} u\ \nabla \cdot \zeta\, d\mathcal{L}^n = \int_{\mathbb{R}^n} \zeta \cdot \sigma\ d|Du| = \int_{\mathbb{R}^n} \zeta \cdot\ dDu \tag{9}$$

(We have just combined and earlier identities, with partially new notation.)

We will use the notation

$$\|u\|_{BV} := \int_{\mathbb{R}^n} |u| + |Du|(\mathbb{R}^n).$$

We collect some basic facts about BV functions.

Lemma 2. *If $u \in C^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} |\nabla u| dx < \infty$, then $u \in BV(\mathbb{R}^n)$ and $Du = \mathcal{L}^n \llcorner \nabla u$.*

PROOF. This follows directly by conparing (9) with the formula

$$-\int_{\mathbb{R}^n} u \,\nabla\cdot\zeta \,d\mathcal{L}^n = \int_{\mathbb{R}^n} \zeta\cdot\nabla u \; d\mathcal{L}^n$$

for $\zeta \in C^1_c(\mathbb{R}^n,\mathbb{R}^n)$, which follows for $u \in C^1$ from the divergence theorem. □

The previous lemma implies that if $u \in C^1$, then $|Du| = \mathcal{L}^n \llcorner |\nabla u|$ and that $\sigma(x) = \frac{\nabla u}{|\nabla u|}(x)$. Although σ is undefined where $\nabla u = 0$, this does not concern us, since the set $\{x \in \mathbb{R}^n : \nabla u(x) = 0\}$ has $|Du|$-measure 0, and $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is only required to be $|Du|$-measurable.

The next result is rather straightforward exercise (especially if you have ever seen any similar arguments, which you should have seen by now....)

Lemma 3. *weak lower semicontinuity: Assume that (u_ℓ) is a sequence of BV functions, and that*

$$\sup_\ell |Du_\ell|(\mathbb{R}^n) < \infty$$

and that $u_\ell \to u$ in L^1. Then

$$Du_\ell \to Du \text{ weakly as measures, and}$$
$$|Du|(\mathbb{R}^n) \le \liminf_\ell |Du_\ell|(\mathbb{R}^n)$$

Lemma 4. *weak density of smooth functions. Assume that $u \in BV(\mathbb{R}^n)$, and let $u_\varepsilon := \psi_\varepsilon * u$, where $\psi_\varepsilon(x) = \frac{1}{\varepsilon^n}\psi(\frac{x}{\varepsilon})$ for some fixed $\psi \in C^\infty_c(\mathbb{R}^n)$ such that*

$$\textit{supp}(\psi) \subset B(0,1), \qquad \psi \ge 0, \qquad \int_{\mathbb{R}^n} \psi = 1.$$

Then $u_\varepsilon \to u$ in $L^1(\mathbb{R}^n)$, and

$$\left.\begin{array}{r} Du_\varepsilon \rightharpoonup Du \\ |Du_\varepsilon| \rightharpoonup |Du| \end{array}\right\} \quad \text{weakly as measures,} \qquad \text{and} \qquad |Du_\varepsilon|(\mathbb{R}^n) \to |Du|(\mathbb{R}^n).$$

PROOF. It is a standard fact that if $u \in L^1(\mathbb{R}^n)$ then $\psi_\varepsilon * u \to u$ in L^1 as $\varepsilon \to 0$. (This can be deduced for example from the density of continuous functions in L^1, which is a consequence of Lusin's Theorem.)

Then in view of the above lemma, we already know that $Du_\varepsilon \to Du$ weakly as measures, and also that

$$|Du|(O) \le \liminf_{\varepsilon\to 0} |Du_\varepsilon|(O) \qquad \text{for all open } O \subset \mathbb{R}^n. \tag{10}$$

So to prove that $|Du_\varepsilon| \rightharpoonup |Du|$ weakly as measures, we only need to show that

$$|Du|(F) \ge \limsup_{\varepsilon\to 0} |Du_\varepsilon|(F) \qquad \text{for all closed } F \subset \mathbb{R}^n.$$

This will follow once we verify that

$$|Du|(\mathbb{R}^n) = \lim_{\varepsilon\to 0} |Du_\varepsilon|(\mathbb{R}^n), \tag{11}$$

since then if F is closed, writing $O := \mathbb{R}^n \setminus F$, we have

$$\begin{aligned} |Du|(F) = |Du|(\mathbb{R}^n) - |Du|(O) &\ge \lim_\varepsilon |Du_\varepsilon|(\mathbb{R}^n) - \liminf_\varepsilon |Du_\varepsilon|(O) \\ &= \limsup_\varepsilon |Du_\varepsilon|(F). \end{aligned}$$

So to complete the proof, we must only check (11).

To do this, let $\zeta \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$ be any vector field with compact support in $\mathbb{R}^n$, and such that $\|\zeta\|_\infty \le 1$. Then for any $\varepsilon > 0$, using basic properties of convolutions,

$$\begin{aligned}\int u_\varepsilon \nabla\cdot\zeta \, d\mathcal{L}^n &= \int \psi^\varepsilon * u \nabla\cdot\zeta \, d\mathcal{L}^n \\ &= \int u\psi^\varepsilon * (\nabla\cdot\zeta)\, d\mathcal{L}^n \\ &= \int u \; \nabla\cdot(\psi^\varepsilon * \zeta)\, d\mathcal{L}^n.\end{aligned}$$

It follows from basic properties of convolutions that $\|\psi^\varepsilon * \zeta\|_\infty \le \|\zeta\|_\infty \le 1$, and hence that the right-hand side is bounded by $|Du|(\mathbb{R}^n)$. Since this is true for all ζ, we conclude that

$$|Du_\varepsilon|(\mathbb{R}^n) \le |Du|(\mathbb{R}^n) \qquad \text{for every } \varepsilon > 0.$$

By combining this with (10) (for $O = \mathbb{R}^n$) we obtain (11).

□

Lemma 5. *if $u \in BV(\mathbb{R}^n)$ and $v \in \mathbb{R}^n$, then*

$$\|\tau_v u - u\|_{L^1(\mathbb{R}^n)} \le |v| \; |Du|(\mathbb{R}^n), \qquad \textit{for } \tau_v u(x) := u(x-v) \tag{12}$$

As a result, if (ψ_ε) is a standard mollifier, then

$$\|\psi_\varepsilon * u - u\|_{L^1} \le \varepsilon \; |Du|(\mathbb{R}^n). \tag{13}$$

Proof. In view of Lemma 4, it suffices to prove (12) for $u \in C^1(\mathbb{R}^n)\cap BV(\mathbb{R}^n)$. (This is an exercise – see below.)

And if $u \in C^1 \cap BV$, then

$$\begin{aligned}\int_{\mathbb{R}^n} |u(x-v) - u(x)| &= \int_{\mathbb{R}^n} |\int_0^1 \frac{d}{ds} u(x-sv)\; ds| dx \\ &= \int_{\mathbb{R}^n} |\int_0^1 \nabla u(x-sv)\cdot v \; ds|dx \\ le|v| &\int_0^1 \int_{\mathbb{R}^n} |\nabla u(x-sv)||dx\, ds \\ &= |v| \int_{\mathbb{R}^n} |\nabla u| \\ &= |v| \; |Du|(\mathbb{R}^n).\end{aligned}$$

It follows rather directly that

$$\|\psi_\varepsilon * u - u\|_{L^1} \le \int_{B(0,\varepsilon)} \int_{\mathbb{R}^n} \; \psi_\varepsilon(y) \; |u(x-y) - u(x)| \; dx \; dy \;\; \le \varepsilon |Du|(\mathbb{R}^n).$$

□

Exercise 3. Show that if (12) holds for every $u \in C^1(\mathbb{R}^n)\cap BV(\mathbb{R}^n)$, then in fact it holds for every $u \in BV(\mathbb{R}^n)$.

Lemma 6. *compactness. Assume that (u_ℓ) is a sequence of BV functions and that*

$$\sup_\ell \|u_\ell\|_{BV} < \infty.$$

Then there exists a subsequence ℓ' that converges to a limit u in $L^1_{loc}(\mathbb{R}^n)$. That is, for every bounded open W, $\|u_{\ell'} - u\|_{L(W)} \to 0$ as $\ell' \to \infty$..

PROOF. For every ℓ, let $u_{\ell,\varepsilon} := \psi_\varepsilon * u_\ell$.

It is a standard fact that $u_{\ell,\varepsilon}$ is smooth, and

$$\|\nabla u_{\ell,\varepsilon}\|_\infty = \|\nabla \psi_\varepsilon * u_\ell\|_\infty \le \|\nabla\psi_\varepsilon\|_\infty \|u_\ell\|_1 \le C\varepsilon^{-(n+1)} \sup_\ell \|u\|_{BV}.$$

Similarly

$$\|u_{\ell,\varepsilon}\|_\infty = \|\psi_\varepsilon * u_\ell\|_\infty \le \|\psi_\varepsilon\|_\infty \|u_\ell\|_1 \le C\varepsilon^{-n} \sup_\ell \|u\|_{BV}.$$

Thus for every fixed $\varepsilon > 0$, the sequence $\{u_{\ell,\varepsilon}\}_{\ell=1}^\infty$ is bounded and equicontinuous. By the Arzela-Ascoli Theorem and a diagonalization argument, it follows that there is a subsequence $\{u_{\ell,\varepsilon}\}$ that converges uniformly on compact sets, and and hence also in L^1_{loc}, to a limit $u_{\infty,\varepsilon}$.

Note also that for every bounded open set W, using (13),

$$\begin{aligned}\|u_{\infty,\varepsilon} - u_{\infty,\delta}\|_{L^1(W)} \le \liminf_{\ell'} \Big(& \|u_{\infty,\varepsilon} - u_{\ell',\varepsilon}\|_{L^1(W)} + \|u_{\ell',\varepsilon} - u_{\ell'}\|_{L^1(W)} \\ & + \|u_{\ell'} - u_{\ell',\delta}\|_{L^1(W)} + \|u_{\ell',\varepsilon} - u_{\infty,\delta}\|_{L^1(W)}\Big) \\ \le (\varepsilon+\delta) \sup_\ell & |Du_\ell|(\mathbb{R}^n).\end{aligned}$$

Since L^1 is complete, it follows that $u := L^1 \lim_{\varepsilon\to 0} u_{\infty,\varepsilon}$ exists and that

$$\|u - u_{\infty,\varepsilon}\|_{L^1(\mathbb{R}^n)} = \sup_{W \text{ bounded, open}} \|u - u_{\infty,\varepsilon}\|_{L^1(W)} \le C\varepsilon.$$

Then for every bounded open W and every $\varepsilon > 0$,

$$\begin{aligned}\limsup_{\ell'\to\infty} \|u - u_{\ell'}\|_{L^1(W)} \le \limsup_{\ell'\to\infty} \Big(& \|u - u_{\infty,\varepsilon}\|_{L^1(W)} + \|u_{\infty,\varepsilon} - u_{\ell',\varepsilon}\|_{L^1(W)} \\ & + \|u_{\ell',\varepsilon} - u_{\ell'}\|_{L^1(W)}\Big) \qquad \le C\varepsilon\end{aligned}$$

where we have again used (13). Since ε is arbitrary, this completes the proof. $\square$

For $u \in BV(\mathbb{R}^n)$, we will use the notation

$$MDu(x) := \sup_{r>0} \frac{|Du|(B(x,r))}{\alpha_n r^n}$$

where $\alpha_n := \mathcal{L}^n(B(0,1))$. Thus, MDu is the maximal function associated to the total variation measure $|Du|$.

It follows from rather standard arguments, using the Vitali covering lemma, that

$$\mathcal{L}^n\big(\{x \in \mathbb{R}^n : MDu(x) < \lambda\}\big) \ \le \ \frac{c_n}{\lambda}|Du|(\mathbb{R}^n).$$

In particular, $MDu < \infty$ $\mathcal{L}^n$ almost everywhere.

We will need the following fact.

Exercise 4. Assume that $u \in BV(\mathbb{R}^n)$, and define $u_\varepsilon := \psi_\varepsilon * u$ as in Lemma 4. Prove that

$$MDu_\varepsilon(x) \to MDu(x) \quad \text{ as } \varepsilon \to 0, \text{ for } \textit{every } x \in \mathbb{R}^n.$$

Lemma 7. *Assume that $u \in BV(\mathbb{R}^n)$. If x, y are Lebesgue points of u then*

$$|u(x) - u(y)| \le C_n|x-y| \ (MDu(x) - MDu(y)) \tag{14}$$

PROOF. **1**. We claim that if x is a Lebesgue point of u, then

$$\fint_{B(x,r)} \frac{|u(x)-u(y)|}{|x-y|}\,dy \le MDu(x). \tag{15}$$

We first prove this for $u \in BV \cap C^1(\mathbb{R}^n)$. It clearly suffices to prove it for $x = 0$. Thus we compute, using Fubini's Theorem and a change of variables,

$$\begin{aligned}
\fint_{B(0,r)} \frac{|u(0)-u(y)|}{|y|}\,dy &= \frac{1}{\alpha_n r^n}\int_{B(0,r)} \left|\int_0^1 \frac{1}{|y|}\frac{d}{ds}u(sy)\,ds\right| dy \\
&\le \frac{1}{\alpha_n r^n}\int_{B(0,r)}\int_0^1 |\nabla u(sy)|\,ds\,dy \\
&= \int_0^1 \frac{1}{\alpha_n r^n}\frac{1}{s^n}\int_{B(0,rs)} |\nabla u(z)|\,dz\,ds \\
&= \int_0^1 \frac{|Du|(B(0,rs))}{\alpha_n (rs)^n} ds \\
&\le MDu(0).
\end{aligned}$$

This is (15).

For arbitrary $u \in BV$, we define $u_\varepsilon = \psi_\varepsilon * u$ as in Lemma 4. Then (15) applies to every u_ε for every ε. Also, it is rather standard, and not hard to check, that $u_\varepsilon \to u$ at every Lebesgue point of u, so (15) follows from Exercise 4.

2. Next, we define $\theta = \theta(n) \in (0,1)$ by requiring that if x_1, x_2 are any distinct points in $\mathbb{R}^n$, and $r := |x_2 - x_1|$, then

$$\frac{\mathcal{L}^n(B(x_1,r)\cap B(x_2,r))}{\alpha_n r^n} = 3\theta$$

The point is that the numerator depends only on $r := |x_2 - x_1|$ and scales like r^n, so that such a number exists.

Now fix any two Lebesgue points x_1, x_2 of u, and let $A := B(x_1,r)\cap B(x_2,r)$, for $r := |x_2 - x_1|$. We claim that for $i = 1, 2$,

$$\mathcal{L}^n\left(\{z \in A : \frac{|u(x_i)-u(z)|}{|x_i - z|} > \frac{1}{\theta}MDu(x_i)\}\right) < \frac{1}{3}\mathcal{L}^n(A). \tag{16}$$

Indeed, for any $k > 0$, Chebyshev's inequality implies that

$$\begin{aligned}
k\,\mathcal{L}^n\left(\{z \in A : \frac{|u(x_i)-u(z)|}{|x_i - z|} > k\}\right) &< \int_A \frac{|u(x_i)-u(z)|}{|x_i-z|}dz \\
&\le \int_{B(x_i,r)} \frac{|u(x_i)-u(z)|}{|x_i-z|}dz \\
&\le \alpha_n r^n MDu(x_i) \\
&= \frac{\mathcal{L}^n(A)}{3\theta}MDu(x_i)
\end{aligned}$$

Setting $k = \frac{1}{\theta}MDu(x_i)$, we deduce (16).

3. It follows from (16) that there exists $z \in A$ such that if we define $C_n := \frac{1}{\theta}$, then

$$\frac{|u(x_i)-u(z)|}{|x_i - z|} \le C_n MDu(x_i) \quad \text{for } i = 1, 2$$

Since $|x_i - z| \le r := |x_1 - x_2$ for $z \in A$, it follows that

$$|u(x_1) - u(x_2)| \le |u(x_1) - u(z)| + |u(z) - u(x_2)| \le C_n |x_2 - x_1| (MDu(x_1) + MDu(x_2)).$$

□

Corollary 1. *If $u \in BV(\mathbb{R}^n)$, then the set $\{(x, u(x)) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{n+1}$ (that is, the graph of u) is countably n-rectifiable.*

Exercise 5. Supply the (short) proof.

Finally we prove

Lemma 8. *Assume that μ is a signed Radon measure on $\mathbb{R}^n$ with finite total variation, and that*

$$\int_{\mathbb{R}^n} \nabla \cdot \zeta \, d\mu \ \le C\|\zeta\|_\infty \qquad \text{for all } \zeta \in C^1_c(\mathbb{R}^n, \mathbb{R}^n).$$

Then $\mu \ll \mathcal{L}^n$, and $\frac{d\mu}{d\mathcal{L}^n}$ is a BV function.

As a consequence, if L is a bounded linear functional on $C^\infty_c(\mathbb{R}^n)$ such that

$$|L(\phi)| \le C\|\phi\|_\infty \text{ for all } \phi \in C_c(\mathbb{R}^n), \qquad |L(\nabla \cdot \zeta)| \le C\|\zeta\|_\infty \text{ for all } \zeta \in C^1_c(\mathbb{R}^n),$$

then there is a function $u \in BV(\mathbb{R}^n)$ such that $L(\phi) = \int u\phi \, dx$ for all $\phi \in C_c(\mathbb{R}^n)$.

Proof. Let $u_\varepsilon := \psi_\varepsilon * \mu$, where ψ_ε is a standard mollifier, so that

$$u_\varepsilon(x) := \int \psi_\varepsilon(x - y) \, d\mu(y).$$

Then it is straightforward to check that u_ε is a BV function, and moreover (arguing as in the proof of (11) in Lemma 4) that

$$\|u_\varepsilon\|_{L^1} \le |\mu|(\mathbb{R}^n), \quad |Du_\varepsilon|(\mathbb{R}^n) \le \sup\{\int \nabla \cdot \zeta \, d\mu : \zeta \in C^1_c(\mathbb{R}^n, \mathbb{R}^n), \|\zeta\|_\infty \le 1\}$$

for all $\varepsilon > 0$. Thus there exists a subseqnece ε' and a function $u \in BV$ such that $u_{\varepsilon'} \to u$ in $L^1_{loc}(\mathbb{R}^n)$. Then for any $\phi \in C_c(\mathbb{R}^n)$,

$$\begin{aligned}
\int_{\mathbb{R}^n} u \ \phi \ d\mathcal{L}^n &= \lim_{\varepsilon' \to 0} \int_{\mathbb{R}^n} u_{\varepsilon'} \ \phi \ d\mathcal{L}^n \\
&= \lim_{\varepsilon' \to 0} \int_{\mathbb{R}^n} \psi_{\varepsilon'} * \phi d \ mu \\
&= \int_{\mathbb{R}^n} \phi d \ mu.
\end{aligned}$$

This implies that $\mu \ll \mathcal{L}^n$, and then it follows that $u \in BV$. The final conclusion of the lemma then is a consequence of the Riesz Representation Theorem. □