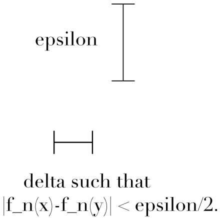
We are proving the Arzela-Ascoli Theorem.

So far we have shown that given a bounded sequence in $C^0([a, b])$, we can find a subsequence, say (f_n), such that

$$f_n(x) \to \text{ a limit } f(x) \text{for every } x \text{ in a countable dense subset of } [a, b]. \tag{1}$$

We hope to use equicontinuity to upgrade this to uniform convergence.

That is, to complete the proof of the theorem, it suffices to prove thatif (f_n) is an equicontinuous sequence that satisfies (1), then f_n converges uniformly to a limit.


epsilon
delta such that
|f_n(x)-f_n(y)| < epsilon/2.

epsilon

Suppose • f(x)= z

delta such that
|f_n(x)-f_n(y)| < epsilon/2.

epsilon

delta such that
|f_n(x)-f_n(y)| < epsilon/2.

Suppose f(x)= z

and |f_n(x) -z| < epsilon/2

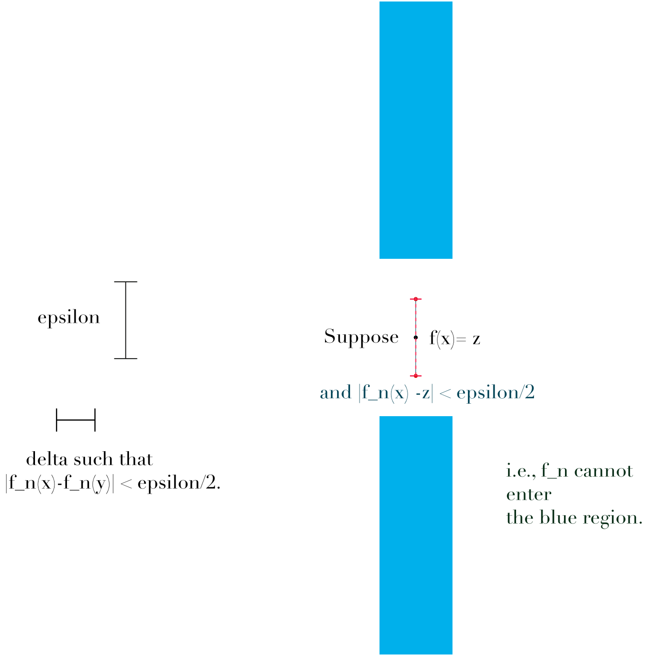
epsilon

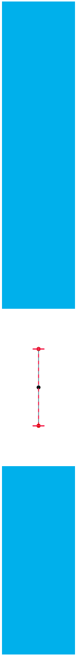
Suppose f(x)= z

and |f_n(x) -z| < epsilon/2

delta such that
|f_n(x)-f_n(y)| < epsilon/2.

if |y-x|<delta, it follows that
|f_n(y)-z|<epsilon

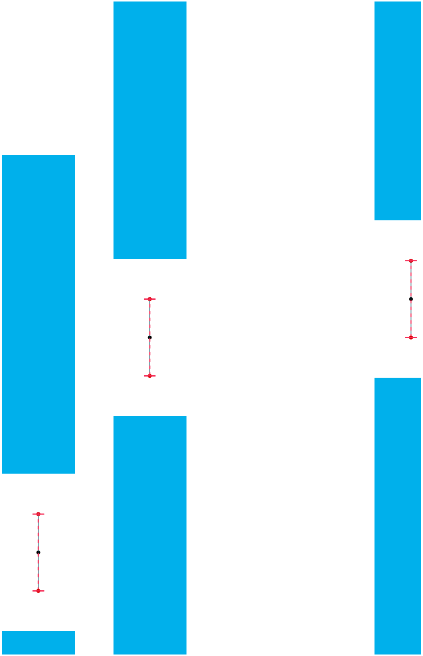

epsilon
Suppose
f(x)= z
and |f_n(x) -z| < epsilon/2
delta such that
|f_n(x)-f_n(y)| < epsilon/2.
i.e., f_n cannot
enter
the blue region.

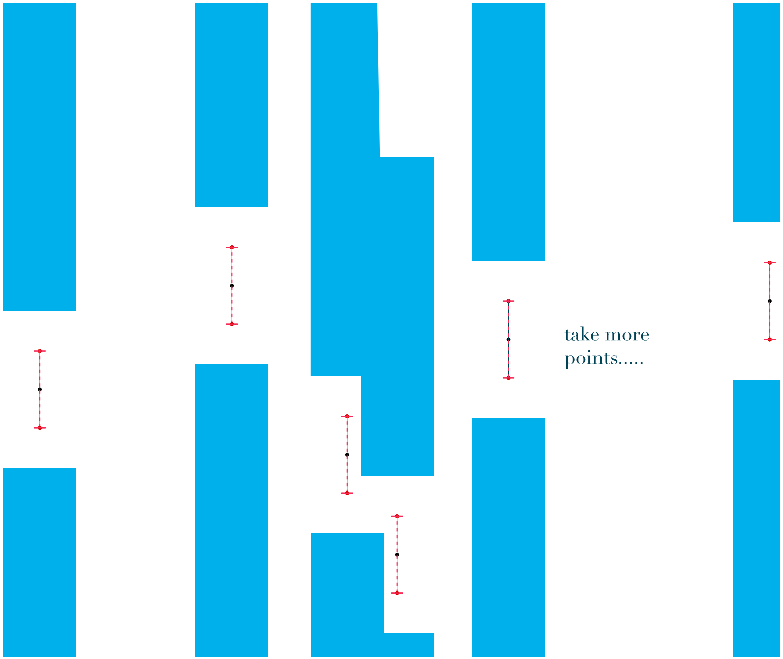


now choose points
x_1, x_2, ... , x_m,
and choose N so large
that

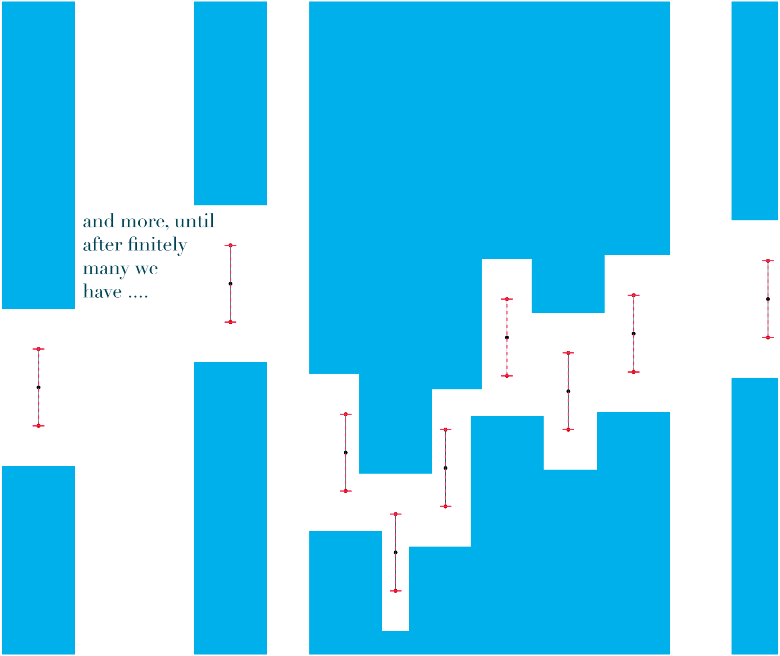
|f_n(x_k) - f(x_k)|< epsilon/2

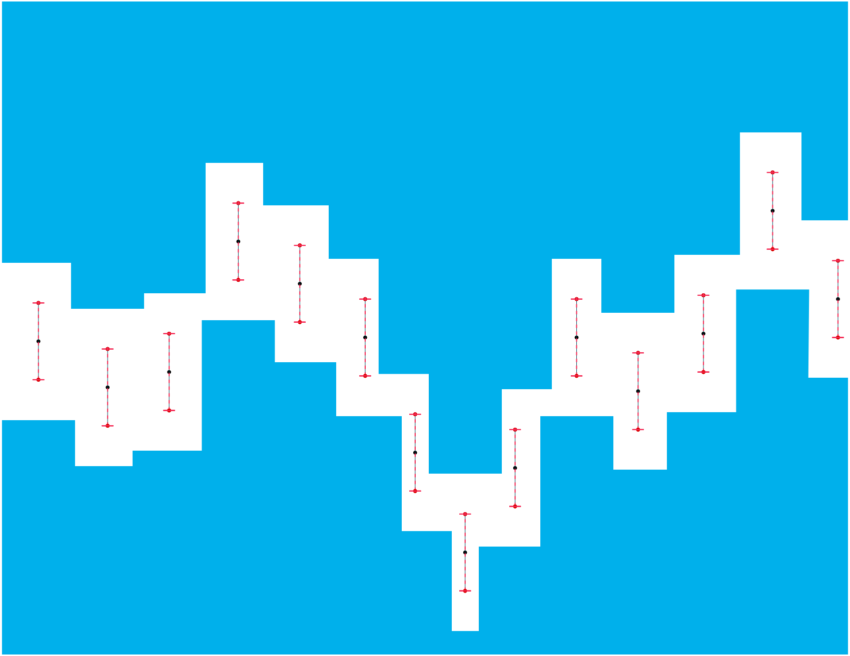
for all k =1,...,m,
whenever n>N.




take more
points.....

and more, until after finitely many we have ....





Actually, in the above arugment, while getting carried away with the graphics, I sort of screwed up my epsilons. In the final picture, we have not shown that

$$|f_n(y) - f(y)| < \varepsilon \qquad \text{whenever } n \geq N.$$

The argument shows however that for all $n \geq N$, any f_n must be confined to the blue tube, which has height at most 2ε. It thus implies that (f_n) is a Cauchy sequence with respect to the sup norm, or equivalently, a Cauchy sequence in the metric space C^0, and this implies the existence of a function f such that $f_n \to f$ uniformly as $n \to \infty$.