MATC37 Assignment 1, due Jan 24

1. Open and closed sets I

Recall that a set $E \subseteq \mathbb{R}^d$ is called open if for every $x \in E$ there exists an $r > 0$ such that $B_r(x) \subseteq E$. Also recall that a set $E \subseteq \mathbb{R}^d$ is by definition closed if E^c is open.

i. Explain why $\mathbb{R}^d$ is both open and closed.

ii. Prove that the unit cube $Q := \{x \in \mathbb{R}^d \, : \, 0 \leq x_1 \leq 1, \ldots, 0 \leq x_d \leq 1\}$ is closed.

2. Open and closed sets II

i. Prove that if $E_1, \ldots, E_k \subseteq \mathbb{R}^d$ are closed sets, then their union $E_1 \cup \ldots \cup E_k$ is also closed.

ii. Give an example of a countable collection of open sets $U_1, U_2, \ldots \subseteq \mathbb{R}$ such that their intersection $\bigcap_{i=1}^{\infty} U_i$ is not closed.

3. Closure, interior, and boundary

Recall from the lecture or textbook that for a set $E \subseteq \mathbb{R}^d$ we have the notions of closure $\bar{E}$, interior E°, and boundary $\partial E = \bar{E} \setminus E^\circ$.

i. Consider the annular region $A := \{x \in \mathbb{R}^2 : 2 < |x| \leq 5\} \subseteq \mathbb{R}^2$. Find $\bar{A}$, A°, and ∂A.

ii. Consider the set of rational numbers $\mathbb{Q} \subseteq \mathbb{R}$. Find $\bar{\mathbb{Q}}$, $\mathbb{Q}^\circ$, and $\partial \mathbb{Q}$.

4. The Cantor set

Recall that the Cantor set $C \subseteq \mathbb{R}$ is defined as

$$C := \bigcap_{k=0}^{\infty} C_k,$$

where the sequence C_k is given by iteratively removing middle thirds from the unit interval, i.e.

$$\begin{aligned}
&C_0 = [0, 1] \\
&C_1 = [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1] \\
&C_2 = [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1] \\
&\ldots
\end{aligned}$$

Prove that C is compact and uncountable.

5. Distance between subsets of Euclidean space

Recall from the lecture that we defined the distance between (nonempty) sets $E, F \subseteq \mathbb{R}^d$ by

$$d(E, F) := \inf_{x \in E, y \in F} |x - y| \, .$$

Prove that if E and F are compact, then there exist $\hat{x} \in E$ and $\hat{y} \in F$ such that

$$d(E, F) = |\hat{x} - \hat{y}| \, .$$

6. Compactness and continuity

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous function, and let $C \subset \mathbb{R}^d$ be compact. Prove that the image $f(C)$ is also compact.

7. The space of continuous functions on the unit interval

Let $\mathcal{C} = \{f : [0,1] \to \mathbb{R} \,|\, f \text{ continuous }\}$ be the space of continuous functions on the unit interval and let

$$d(f,g) := \int_0^1 |f(x) - g(x)|\, dx \qquad\qquad (f, g \in \mathcal{C}).$$

Prove that d is a metric, i.e. prove that it is positive definite, symmetric, and satisfies the triangle inequality.

Please feel free to discuss the homework problems among yourselves and with me. But write up your assignments in your own words, and be ready to defend them! Your work will be judged on the clarity of your presentation as well as correctness and completeness.

We will randomly select 3 questions, for which you will receive points $p_1, p_2, p_3 \in \{0, 1, 2, 3\}$ depending on how well you solved them. Let s be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1 + p_2 + p_3 - s, 0) \in \{0, 1, \ldots, 9\}$.