NONLINEAR LAPLACE EQUATION

LECTURE NOTES FOR MARCH 16

1. Dirichlet problem for the Laplace equation

In prior lectures we discussed the Dirichlet problem for the Laplace equation:

$$\begin{cases} \Delta u = 0 & \text{in } D, \\ u = h & \text{on } \partial D. \end{cases} \tag{1.1}$$

Here, $D \subset \mathbb{R}^3$ is the given domain (e.g. a ball or a cube), and h is a given function defined on the boundary ∂D.

In particular, we learned that the solution of the problem (1.1) is unique. Indeed, suppose u_1 and u_2 are two solutions of the problem (1.1). Then, their difference $v := u_1 - u_2$ solves

$$\begin{cases} \Delta v = 0 & \text{in } D, \\ v = 0 & \text{on } \partial D. \end{cases} \tag{1.2}$$

Thus, the maximum principle implies that $v \equiv 0$ in D, i.e. $u_1 \equiv u_2$.

Instead of the maximum principle we could use the energy method: Multiplying the equation $\Delta v = 0$ by v and integrating we get $\int_D v\Delta v = 0$ and thus via integration by parts $\int_D |\nabla v|^2 = 0$. Hence, v is constant, and since $v = 0$ on ∂D this constant must be zero, i.e. $v \equiv 0$.

In particular, the energy method tells us that the energy

$$E[u] = \tfrac{1}{2} \int_D |\nabla u|^2 \tag{1.3}$$

plays an important role in the analysis of the problem (1.1). Physically, we can think of (1.3) as the energy of our system. Now, it is a general principle in physics that any system prefers to go to the state of lowest energy, called the ground state. We can think of the solution of (1.1) as the ground state of our system. In fact, we have:

Theorem 1.4 (Dirichlet principle)**.** *Let u be the unique solution of the Dirichlet problem* (1.1)*. If w is any other function in D satisfying $w = h$ on ∂D, then*

$$E[w] \geq E[u]. \tag{1.5}$$

In other words, our harmonic function u minimizes the energy among all functions (with the same boundary condition).

Proof. Let u and w be as in the statement of the theorem. Consider the difference $v := u - w$. Then, we can compute

$$E[w] = \tfrac{1}{2}\int_D |\nabla(u-v)|^2 \tag{1.6}$$

$$= \tfrac{1}{2}\int_D |\nabla u|^2 - \int_D \nabla u \cdot \nabla v + \tfrac{1}{2}\int_D |\nabla v|^2. \tag{1.7}$$

Now, using $\Delta u = 0$ in D and $v = 0$ on ∂D we see that

$$-\int_D \nabla u \cdot \nabla v = \int_D v\Delta u - \int_{\partial D} v\frac{\partial u}{\partial n} = 0. \tag{1.8}$$

Together with the obvious inequality $\int_D |\nabla v|^2 \geq 0$ we conclude that

$$E[w] \geq \tfrac{1}{2}\int_D |\nabla u|^2 = E[u]. \tag{1.9}$$

This proves the theorem. □

2. Dirichlet problem for the nonlinear Laplace equation

Let us now consider the Dirichlet problem for the nonlinear Laplace equation:

$$\begin{cases} \Delta u + u^p = 0 & \text{in } D, \\ u = h & \text{on } \partial D. \end{cases} \tag{2.1}$$

Here, $p > 1$ is a given number. Note that $\Delta u + u^p = 0$ is a nonlinear equation, i.e. the sum of two solutions is not a solution in general. This nonlinearity gives rise to new interesting phenomena in stark contrast to the linear analysis from the previous section. In particular, we will see that the behaviour of (2.1) depends crucially on the exponent p.

Example 2.2. Suppose $D = B_1(0)$ is the unit ball, $p = 5$, and $h \equiv \sqrt{\sqrt{3}/2}$. Let us look for a radial solution $u = u(r)$. Recalling the formula for the Laplacian in spherical coordinates, we thus have to solve the ODE

$$u_{rr} + \frac{2}{r}u_r + u^5 = 0. \tag{2.3}$$

Let us make the ansatz[1]

$$u = \frac{c}{\sqrt{1+r^2}}. \tag{2.4}$$

Differentiating gives

$$u_r = -\frac{cr}{(1+r^2)^{3/2}}, \tag{2.5}$$

and

$$u_{rr} = \frac{3cr^2}{(1+r^2)^{5/2}} - \frac{c}{(1+r^2)^{3/2}}. \tag{2.6}$$

Thus,

$$u_{rr} + \frac{2}{r}u_r + u^5 = \frac{3cr^2}{(1+r^2)^{5/2}} - \frac{3c}{(1+r^2)^{3/2}} + \frac{c^5}{(1+r^2)^{5/2}} \tag{2.7}$$

$$= \frac{-3c + c^5}{(1+r^2)^{5/2}}. \tag{2.8}$$

Hence, choosing $c = 3^{1/4}$ we have found a solution. □

Associated to the problem (2.1) we have the energy functional

$$E[u] = \tfrac{1}{2}\int_D |\nabla u|^2 - \tfrac{1}{p+1}\int_D |u|^{p+1}. \tag{2.9}$$

For p close to 1 the term $\int_D |\nabla u|^2$ dominates. For p very large the term $\int_D |u|^{p+1}$ dominates. The transition happens at the critical exponent $p = 5$ that we have encountered in our example. In fact:

Theorem 2.10. *The problem*

$$\begin{cases} \Delta u + u^p = 0 & \text{in } B, \\ u = 0 & \text{on } \partial B. \end{cases} \tag{2.11}$$

has nontrivial solutions for $1 < p < 5$*, but only the zero solution for* $p > 5$.

Sketch of proof. If $1 < p < 5$ one can find nontrivial solutions via the mountain pass method (we skip this, but if you are interested you can read it in Evans in Section 8.5).
Suppose now $p > 5$. Multiplying our equation by u and integrating by parts we obtain

$$\int_B |\nabla u|^2 = \int_B |u|^{p+1}. \tag{2.12}$$

[1]Note that for r large the function u approaches $v = \frac{c}{r}$ which solves $v_{rr} + \frac{2}{r}v_r = 0$.

On the other hand, multiplying our equation by $x \cdot \nabla u$ and integrating we obtain

$$\int_B \Delta u \, x \cdot \nabla u + \int_B u^p \, x \cdot \nabla u = 0. \tag{2.13}$$

After integration by parts (we also skip this remarkable computation, but if you are interested you can read it in Evans in Section 9.4) this can be rewritten as

$$\int_B |\nabla u|^2 + \int_{\partial B} |x||\nabla u|^2 = \frac{6}{p+1} \int_B |u|^{p+1}. \tag{2.14}$$

Subtracting (2.12) from (2.14) we obtain

$$\left(\frac{6}{p+1} - 1 \right) \int_B |u|^{p+1} \geq 0. \tag{2.15}$$

Since $p > 5$ we have $\frac{6}{p+1} - 1 < 0$, hence $u \equiv 0$. □