Real Analysis 1, Assignment 1, due Sep 21

1. Push-forward of a σ -algebra

Let (X, \mathcal{M}) be a measurable space, and let $f: X \to Y$ be a map. Consider

$$\mathcal{M}_f := \{ B \subset Y \mid f^{-1}(B) \in \mathcal{M} \} \,. \tag{1}$$

Prove that (Y, \mathcal{M}_f) is a measurable space.

2. Borel σ -algebra on \mathbb{R} (c.f. Folland Prop. 1.12)

Prove that the Borel σ -algebra on \mathbb{R} is generated by intervals of the form (a, ∞) , i.e. prove that

$$\mathcal{B}_{\mathbb{R}} = \sigma(\{(a,\infty) \mid a \in \mathbb{R}\}).$$

3. Inclusion-Exclusion (Folland 1.9)

Let (X, \mathcal{M}, μ) be a measure space and $A, B \in \mathcal{M}$. Prove that

$$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B).$$

4. ν -measurablility

Let (X, \mathcal{M}, μ) be a measure space and consider the induced outer measure

$$\nu(A) := \inf\{\mu(B) \mid B \supset A, B \in \mathcal{M}\}.$$

Prove that every $A \in \mathcal{M}$ is ν -measurable.

5. Classification of $\{0,1\}$ -valued measures on \mathbb{R}

Let $\mu : \mathcal{P}(\mathbb{R}) \to \{0,1\}$ be a $\{0,1\}$ -valued measures on \mathbb{R} . Prove that either $\mu = 0$ or $\mu = \delta_{x_0}$ for some $x_0 \in \mathbb{R}$. (Hint: Recall the proof of the Bolzano-Weierstrass theorem.)

Please feel free to discuss the homework problems among yourselves and with me and the TA. But write up your assignments in your own words, and be ready to defend them! Your work will be judged on the clarity of your presentation as well as correctness and completeness.

The TA will randomly select 2 questions, for which you will receive points $p_1, p_2 \in \{0, 1, 2, 3\}$ depending on how well you solved them. Let s be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1 + p_2 - s, 0) \in \{0, 1, \dots, 6\}$.