Real Analysis 1, Assignment 10, due Nov 30

1. Hardy-Littlewood maximal function (Stein-Shakarchi Exer 3.4)

(a) Prove that if $f \in L^1(\mathbb{R}^n)$ is not zero, then $Mf(x) \geq \frac{c}{|x|^n}$ for some $c > 0$ and all $|x| \geq 1$, in particular $Mf \notin L^1(\mathbb{R}^n)$ (Hint: Use the fact that $\int_B |f| > 0$ for some ball B).

(b) Show that the weak-L^1 estimate

$$m(\{x : Mf(x) > \alpha\}) \leq C/\alpha$$

for all $\alpha > 0$ whenever $\int |f| = 1$, is best possible in the following sense: If f is supported in the unit ball with $\int |f| = 1$, then

$$m(\{x : Mf(x) > \alpha\}) \geq c'/\alpha$$

for some $c' > 0$ and all sufficiently small α.

2. Density of a Borel set (Folland Exer 3.25)

If E is a Borel set in $\mathbb{R}^n$, the density $D_E(x)$ of E at x is defined as

$$D_E(x) = \lim_{r \to 0} \frac{m(E \cap B_r(x))}{m(B_r(x))}$$

whenever the limit exists.

(a) Given a number $\alpha \in (0, 1)$, find examples of E and x such that $D_E(x) = \alpha$.

(b) Show that $D_E(x) = 1$ for a.e. $x \in E$ and $D_E(x) = 0$ for a.e. $x \in E^c$.

(c) Give an example of E and x such that $D_E(x)$ does not exist.

3. Measure associated to increasing function (Folland Exer 1.28)

Let $F : \mathbb{R} \to \mathbb{R}$ be increasing and right continuous, and let $\mu_F : \mathcal{B}_{\mathbb{R}} \to [0, \infty]$ be the associated measure satisfying $\mu_F((a, b]) = F(b) - F(a)$. Prove that $\mu_F(\{a\}) = F(a) - F(a-)$, $\mu_F([a, b)) = F(b-) - F(a-)$, $\mu_F([a, b]) = F(b) - F(a-)$, and $\mu_F((a, b)) = F(b-) - F(a)$.

4. Total variation of measures and functions (Folland Exer 3.28)

Let $F \in NBV$, and let $G(x) = |\mu_F|((-\infty, x])$. Prove that $|\mu_F| = \mu_{T_F}$ by showing that $G = T_F$ via the following steps:

(a) From the definition of T_F, show $T_F \leq G$.

(b) Show that $|\mu_F(E)| \leq \mu_{T_F}(E)$ when E is an interval, and hence when E is a Borel set.

(c) $|\mu_F| \leq \mu_{T_F}$, and hence $G \leq T_F$ (Hint: use HW problem 9.1).

5. FTC, integration by parts, and substitution (Folland Exer 1.35, 1.36)

(a) Let $F, G : [a, b] \to \mathbb{R}$ be absolutely continuous. Prove that FG is also absolutely continuous, and that

$$\int_a^b F'G\,dx = F(b)G(b) - F(a)G(a) - \int_a^b FG'\,dx\,.$$

(b) Let $G : [a, b] \to \mathbb{R}$ be an absolutely continuous increasing function, and set $c = G(a)$, $d = G(b)$. Prove that if $f : [c, d] \to \mathbb{R}$ is Borel-measurable and integrable, then

$$\int_c^d f(y)dy = \int_a^b f(G(x))G'(x)\,dx\,.$$

Please feel free to discuss the homework problems among yourselves and with me and the TA. But write up your assignments in your own words, and be ready to defend them! Your work will be judged on the clarity of your presentation as well as correctness and completeness.

The TA will randomly select 2 questions, for which you will receive points $p_1, p_2 \in \{0, 1, 2, 3\}$ depending on how well you solved them. Let s be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1 + p_2 - s, 0) \in \{0, 1, \ldots, 6\}$.