Real Analysis 1, Assignment 11, due Dec 7

1. The Sobolev space H^1 for $n = 1$

Let $\Omega = (a, b) \subset \mathbb{R}$ be a bounded open interval. Let $u \in H^1(\Omega)$.

(a) Prove that u is equal a.e. to an absolutely continuous function, and that u' (which exists a.e.) belongs to $L^2(\Omega)$.

(b) Prove that u is $C^{1/2}$-Hölder with the estimate $|u(x) - u(y)| \le ||u||_{H^1} |x - y|^{1/2}$.

2. The Sobolev space H^1 for $n = 2$

Let $\Omega = B_1 \subset \mathbb{R}^2$ be the unit ball.

(a) Prove that if $u \in H^1(\Omega)$, then $u \in L^q(\Omega)$ for all $q < \infty$.

(b) Show that the function $u(x) = \log\log(1 + 1/|x|)$ is in $H^1(\Omega)$ but not in $L^\infty(\Omega)$.

3. Weak derivative of products and convolutions

Let $u \in H^1(\mathbb{R}^n)$ and $\psi \in C_c^\infty(\mathbb{R}^n)$.

(a) Prove that $\psi u \in H^1(\mathbb{R}^n)$ and $\nabla(\psi u) = \psi \nabla u + u \nabla \psi$.

(b) Prove that $\psi * u \in H^1(\mathbb{R}^n)$ and $\nabla(\psi * u) = \psi * \nabla u = \nabla \psi * u$.

4. Poincare inequality

Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected open set with C^1 boundary. Prove that there exists a constant $C = C(n, \Omega) < \infty$, such that

$$||u - \bar{u}||_{L^2(\Omega)} \le C ||\nabla u||_{L^2(\Omega)}$$

for all $u \in H^1(\Omega)$, where $\bar{u} = \frac{1}{|\Omega|} \int_\Omega u$ denotes the average of u. (Hint: Argue by contradiction and use the Rellich-Kondrachov theorem.)

5. Ground state energy for the Schrödinger equation (c.f. Lieb-Loss Sec. 11)

Let $V \in L^{3/2}(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, and assume that V vanishes at infinity, i.e.

$$|\{x \in \mathbb{R}^3 : |V(x)| > a\}| < \infty \qquad \text{for all } a > 0\,.$$

Consider the energy functional

$$E[\psi] = \int_{\mathbb{R}^3} (|\nabla \psi|^2 + V |\psi|^2)$$

and let

$$E_0 = \inf\{E[\psi] : \psi \in H^1(\mathbb{R}^3) \text{ with } ||\psi||_{L^2(\mathbb{R}^3)} = 1\}\,.$$

(a) Prove that if $V = 0$, then $E_0 = 0$ and there exists no minimizer.

(b) Is it true that for any potential V, we have $E_0 \le 0$?

(c) Prove that if $V(x) = -|x|^{-1}$ is the potential of the Hydrogen atom, then $E_0 \leq -1/4$. (Hint: Consider the function $\psi(x) = \exp(-|x|/2)$.)

(d) Try to find some general conditions for the potential V, which imply that $E_0 < 0$.

Please feel free to discuss the homework problems among yourselves and with me and the TA. But write up your assignments in your own words, and be ready to defend them! Your work will be judged on the clarity of your presentation as well as correctness and completeness.

The TA will randomly select 2 questions, for which you will receive points $p_1, p_2 \in \{0, 1, 2, 3\}$ depending on how well you solved them. Let s be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1 + p_2 - s, 0) \in \{0, 1, \ldots, 6\}$.