Real Analysis 1, Assignment 2, due Sep 28

1. Scaling properties of the Lebesgue-measure (Stein-Shakarchi Exer 1.7)

If $\delta = (\delta_1, \ldots, \delta_n)$ is an n-tupel of positive real numbers, and $A \subset \mathbb{R}^n$, we define $\delta \cdot A$ by

$$\delta \cdot A = \{(\delta_1 x_1, \ldots, \delta_n x_n) \,|\, (x_1, \ldots, x_n) \in A\}\,.$$

Prove that A is Lebesgue-measurable if and only if $\delta \cdot A$ is Lebesgue-measurable, and that

$$\mu(\delta \cdot A) = \delta_1 \ldots \delta_n \mu(A) \qquad (A \in \mathcal{L})\,.$$

2. The Cantor set

Let $C := \cap_{k=0}^{\infty} C_k \subset [0,1]$ be the Cantor set, which is obtained by iteratively removing the middle third of intervals, i.e. $C_0 = [0,1]$, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, etc. Prove that C has the same cardinality as $\mathbb{R}$, and that C is a Lebesgue null-set. In addition, try to construct a Cantor-like set, by removing pieces smaller than the middle thirds, which has positive Lebesgue-measure.

3. Non-measurable sets (c.f. Folland Exer 1.29)

Consider $\mathbb{R}^n$ with the σ-algebra $\mathcal{L}$ of Lebesgue-measurable sets and the Lebesgue-measure μ. Assume $A \in \mathcal{L}$ satisfies $\mu(A) > 0$. Prove that there exists a subset $B \subset A$ such that $B \notin \mathcal{L}$. (Hint: Reduce to the case $n = 1$ and $A \subset [0,1)$, and consider the decomposition $A = \cup_{r \in \mathbb{Q} \cap [0,1)} A \cap N_r$, where the sets N_r are the ones from the very first lecture.)

4. Lebesgue-measurable but not Borel-measurable

Prove that the Borel σ-algebra $\mathcal{B}_{\mathbb{R}^n}$ is a proper subset of σ-algebra $\mathcal{L}$ of Lebesgue-measurable subsets of $\mathbb{R}^n$. (Hint: Compare the cardinality of the two σ-algebras.)

5. Topological smallness vs. measure theoretic smallness

Consider $\mathbb{R}$ with its standard topology. A subset $A \subset \mathbb{R}$ is called *nowhere dense* if the interior of its closure is empty. Let ν be the outer Lebesgue-measure on $\mathbb{R}$. Prove or disprove:

i) If $A \subset \mathbb{R}$ is dense, then $\nu(A) > 0$.

ii) If $\nu(A) = 0$, then A is nowhere dense.

iii) If $A \subset [0,1]$ and $\nu(A) = 1$, then A is dense in [0,1].

iv) If $A \subset \mathbb{R}$ is nowhere dense, then $\nu(A) = 0$.

Please feel free to discuss the homework problems among yourselves and with me and the TA. But write up your assignments in your own words, and be ready to defend them! Your work will be judged on the clarity of your presentation as well as correctness and completeness.

The TA will randomly select 2 questions, for which you will receive points $p_1, p_2 \in \{0,1,2,3\}$ depending on how well you solved them. Let s be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1 + p_2 - s, 0) \in \{0, 1, \ldots, 6\}$.