# Real Analysis 1, Assignment 3, due Oct 5

## 1. Restriction of outer measure

Let $\nu$ be and outer measure on a topological space $X$, and let $A \subset X$. Define $\nu_{\lfloor A} : \mathcal{P}(X) \to [0, \infty]$ by $(\nu_{\lfloor A})(B) := \nu(A \cap B)$. Prove that:

i) $\nu_{\lfloor A}$ is an outer measure, and each $\nu$-measurable set is $\nu_{\lfloor A}$-measurable.

ii) If $\nu$ is Borel-regular and $A$ is $\nu$-measurable with $\nu(A) < \infty$, then $\nu_{\lfloor A}$ is Borel-regular.

## 2. Lebesgue-measure under linear transformations

Let $(\mathbb{R}^n, \mathcal{L}, \mu)$ be the Lebesgue measure space. In class we proved that for each linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ there exists a constant $c_T \geq 0$ such that $\mu(T(A)) = c_T \mu(A)$, $A \in \mathcal{L}$. Prove that $c_T = |\det T|$. (Hint: Consider the following three types of basic linear transformations: (i) T permutes the standard basis vectors $e_1, \ldots, e_n$, (ii) $Te_1 = \alpha e_1, Te_k = e_k$ for $k = 2, \ldots, n$, (iii) $Te_1 = e_1 + e_2, Te_k = e_k$ for $k = 2, \ldots, n$.)

## 3. Critical values of a continuously differentiable function

Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^1$-function, and let $A := \{x \in \mathbb{R} \,|\, f'(x) = 0\}$. Show that $f(A)$ is a Lebesgue null set. (Hint: Consider the sets $A_{i,\varepsilon} := \{x \in \mathbb{R} \,|\, |x| < n, |f'(x)| < \varepsilon/2^i\}$.)

## 4. Hausdorff measure of the Cantor set (Stein-Shakarchi Exer 7.7)

Let $C \subset [0, 1]$ be the Cantor set. Show that $\mathcal{H}^{\log 2/\log 3}(C) = 1$.

## 5. Sigma algebra on path-space

Consider the space $C([0, 1], \mathbb{R}^n) = \{\gamma : [0, 1] \to \mathbb{R}^n \,|\, \gamma \text{ continuous}\}$ equipped with the metric $d(\gamma_1, \gamma_2) = \max_{t \in [0,1]} |\gamma_1(t) - \gamma_2(t)|$, and let $e_{t_0} : C([0, 1], \mathbb{R}^n) \to \mathbb{R}^n, e_{t_0}(\gamma) = \gamma(t_0)$ be the evaluation map at $t_0 \in [0, 1]$. Prove that

$$\mathcal{B}_{C([0,1], \mathbb{R}^n)} = \sigma(\{e_t^{-1}(U) \,|\, U \subset \mathbb{R}^n \text{ open }, t \in [0, 1]\}).$$

Please feel free to discuss the homework problems among yourselves and with me and the TA. But write up your assignments in your own words, and be ready to defend them! Your work will be judged on the clarity of your presentation as well as correctness and completeness.

The TA will randomly select 2 questions, for which you will receive points $p_1, p_2 \in \{0, 1, 2, 3\}$ depending on how well you solved them. Let $s$ be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1 + p_2 - s, 0) \in \{0, 1, \ldots, 6\}$.