Real Analysis 1, Assignment 4, due Oct 12

1. Hausdorff measure under Lipschitz maps

Let X and Y be metric spaces and $f : X \to Y$ be a Lipschitz map with Lipschitz constant K. Prove that the outer Hausdorff measure satisfies the estimate $\mathcal{H}^\alpha(f(A)) \leq K^\alpha \mathcal{H}^\alpha(A)$.

2. Sequence of measurable functions (Folland Exer 2.1.3)

Let $(X, \mathcal{M})$ be a measurable space. Show that if $f_n : X \to \mathbb{R}$ is a sequence of measurable functions, then $\{x \in X \mid \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is a measurable set.

3. Criterion for measurability of functions (Folland Exer 2.1.4)

Let $(X, \mathcal{M})$ be a measurable space, and let $f : X \to \bar{\mathbb{R}}$. Show that if $f^{-1}((r, \infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then f is measurable.

4. Nonmeasurable composition (Folland Exer 2.1.9)

Contruct an example of a Lebesgue-measurable function F and a continuous function G on $\mathbb{R}$ such that $F \circ G$ is not Lebesgue-measurable. (Hint: Consider the Cantor staircase function.)

5. Derivatives are measurable

Let $f : (a, b) \to \mathbb{R}$ be differentiable. Prove that f' is Borel-measurable.

Please feel free to discuss the homework problems among yourselves and with me and the TA. But write up your assignments in your own words, and be ready to defend them! Your work will be judged on the clarity of your presentation as well as correctness and completeness.

The TA will randomly select 2 questions, for which you will receive points $p_1, p_2 \in \{0, 1, 2, 3\}$ depending on how well you solved them. Let s be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1 + p_2 - s, 0) \in \{0, 1, \ldots, 6\}$.