Real Analysis 1, Assignment 5, due Oct 19

1. Integrable nonnegative functions (Folland Exer 2.12)

Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f \in L^+$ and assume $\int_X f \, d\mu < \infty$. Prove that $\{x \in X \mid f(x) = \infty\}$ is a null set, and that the set $\{x \in X \mid f(x) > 0\}$ is σ-finite.

2. Measure induced by nonnegative function (Folland Exer 2.14)

Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f \in L^+$, and define $\lambda(E) := \int_E f \, d\mu$ for $E \in \mathcal{M}$. Prove that λ is a measure, and that $\int_X g \, d\lambda = \int_X fg \, d\mu$ for every $g \in L^+$. (Hint: First suppose that g is simple.)

3. Recovering monotone convergence (Folland Exer 2.17)

Assume Fatou's lemma and deduce the monotone convergence theorem from it.

4. Limit of integrals (Folland Exer 2.28)

Consider the real line with the Lebesgue measure. Compute the limit

$$\lim_{n \to \infty} \int_a^\infty n(1 + n^2 x^2)^{-1} \, dx$$

and justify your computation. (The answer depends on whether $a > 0$, $a = 0$ or $a < 0$. How does this accord with the various convergence theorems?)

5. Function with nonnegative integrals (Stein-Shakarchi Exer 2.11)

Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f : X \to \mathbb{R}$ be integrable. Prove that if $\int_E f \, d\mu \geq 0$ for all $E \in \mathcal{M}$, then $f(x) \geq 0$ for a.e. x. As a result, if $\int_E f \, d\mu = 0$ for every $E \in \mathcal{M}$, then $f = 0$ a.e.

Please feel free to discuss the homework problems among yourselves and with me and the TA. But write up your assignments in your own words, and be ready to defend them! Your work will be judged on the clarity of your presentation as well as correctness and completeness.

The TA will randomly select 2 questions, for which you will receive points $p_1, p_2 \in \{0, 1, 2, 3\}$ depending on how well you solved them. Let s be the number of questions that you skipped. The total number of points you receive for this assignment is $\max(p_1 + p_2 - s, 0) \in \{0, 1, \ldots, 6\}$.